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Optimal Fractional Fourier Domains for Quadratic Chirps

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The fractional fourier transform can be viewed as a generalization of the Fourier transform. The relation between rotation of a signal in the time-frequency plane to the Fractional Fourier tranform is introduced. In this paper, the fractional Fourier transform and its properties are presented. Further the problem of finding an optimum fractional Fourier Domain, i.e. ..... one in which the energy of a signal is maximally concentrated, is discussed for quadratic chirps. A quadratic chirp is a signal whose frequency bears a quadratic relation in time.

Indexing terms:Time frequency methods, Fractional fourier transform.

1. INTRODUCTION

THE Fractional Fourier Transform (FRFT) has recently been rediscovered and is derived from a generalisation of the fact that the Hermite Gauss functions are eigenfunctions of the conventional fourier transform. That is,

$$\Im\,\{e^{(\frac{-t^2}{2})}H_n(t)\} = e^{(\frac{jn\pi}{2})}\,\{e^{(\frac{w^2}{2})}H_n(w)\} \quad (1)$$

Here, $\Im$ represents the conventional Fourier transform operator; t, w are time and angular frequency respectively while n is an integer. From the above relation it is observed that the eigen value contains integral multiples of $\frac{\pi}{2}$. In an attempt to generalise this operator to yield a general angle, say α, instead of $\frac{\pi}{2}$ in the eigen value of this eigen function, one gets the FRFT operator, defined shortly.

Briefly, the fractional Fourier transform is defined [1,2], with the help of a kernel which depends upon this angle α and is given as;

$$K_\alpha(t,u) = \begin{cases} \sqrt{\dfrac{1-j\cot(\alpha)}{2\pi}}\; e^{j\frac{t^2+u^2}{2}\cot(\alpha) - jut\csc(\alpha)} & \text{if } \alpha \neq n\pi, \\ \delta(t-u) & \text{if } \alpha = 2n\pi, \\ \delta(t+u) & \text{if } \alpha = (2n+1)\pi. \end{cases} \quad (2)$$

where $\delta(t)$ is the ideal impulse function. This angle can be intrepreted as the extent to which we are, either in time or frequency - an angle of zero implies entirely in time, and an angle of 90° implies entirely in frequency. An intermediate angle implies a corresponding joint time-frequency plane[1].

With the above defined kernel, the FRFT at an an α is defined as

$$(F_\alpha f)(u) = \int_{-\infty}^{+\infty} f(t)\,K_\alpha(t,u)\,dt$$

We briefly summarise the properties [1,3] of the FRF

- $F_{2n\pi}$ is the identity transformation, i.e., ... $(F_{2n\pi} f)(u$ $f(u)$.

  Also, $F_{\pi/2}$ is the conventional fourier transform.
- Rotation property:

  $F_{\alpha+\beta} = F_\alpha F_\beta$.
- Linearity:

  $F_\alpha[\,c_1 f(t) + c_{2g}(t)] = c_1 F_\alpha f(t) + c_2\, F_{\alpha g}(t)$.
- From the above, we can conclude that the inve FRFT is theFRFT at an angle $-\alpha$:

  $F_\alpha^{-1}\,(F_\alpha f(t)) = \int_{-\infty}^{+\infty} (F_\alpha f)(u)\,K_{-\alpha}(u,t)\,du$

2. CONCEPT OF TIME-FREQUENCY PLANE

The conventional Fourier transform can interpreted as the projection of a signal's distribution the time-frequency plane, onto the frequency axis. I example, a linear chirp, one whose frequency is a lin

[1]Here, $\delta(x)$ is the ideal impulse function defined by the follow two properties:

$$\delta(x) = 0 \quad \text{for } x \neq 0,$$
$$\int_{-\infty}^{+\infty} \delta(x)\,dx = 1$$

nction of time, distributes along a slant edge in the ne-frequency plane. Note that in the discussion that llows, signals that approximate the echo of a synthetic erture radar (SAR) [4] receiver are considered and nce last for a finite (small) duration. An example of ch a chirp can be seen in Fig 1, where the echo is served from time $t = 20s$ to $t = 50s$.

In the same context, the FRFT at an angle α can be terpreted as a projection [3] of a signal in the time-quency plane on a frequency axis v rotated by an angle A proof of this intuitive idea is presented for linear irps in the Appendix A one can then extend this concept other signals. From the Appendix, it is inferred that the energy of a linear chirp is highly concentrated in the optimum fractional Fourier domain. The idea of concentration on the time-frequency axis, will be clear from the simulation result to follow, in Fig 2.

3. QUADRATIC CHIRPS

The signal under consideration here may be informally described by a parabola in the time-frequency plane, existing from time $t_1 = 40s$ to $t_2 = 70s$ on the time axis, as shown in Fig 3. We seek to examine the projection of this parabola on an axis rotated in such a manner that is perpendicular to the line joining the end points of the curve, as shown in Fig 4. Its projection on a

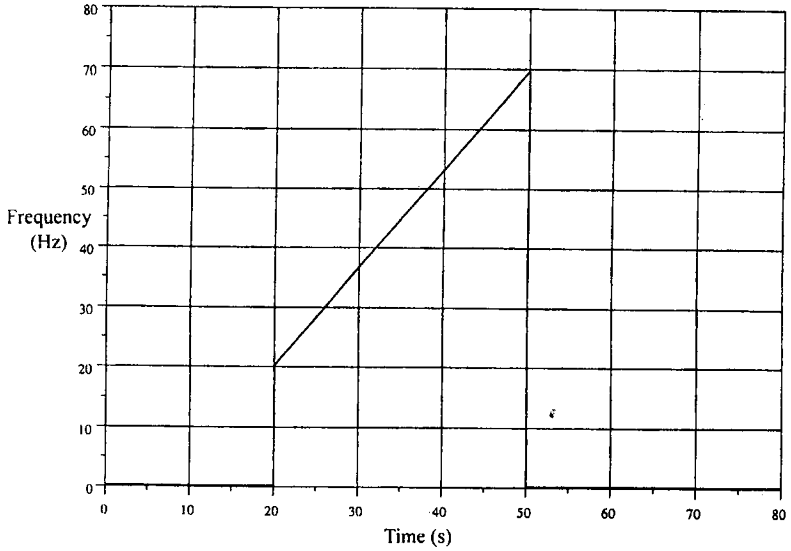


Fig 1 Linear chirp in time-frequency plane

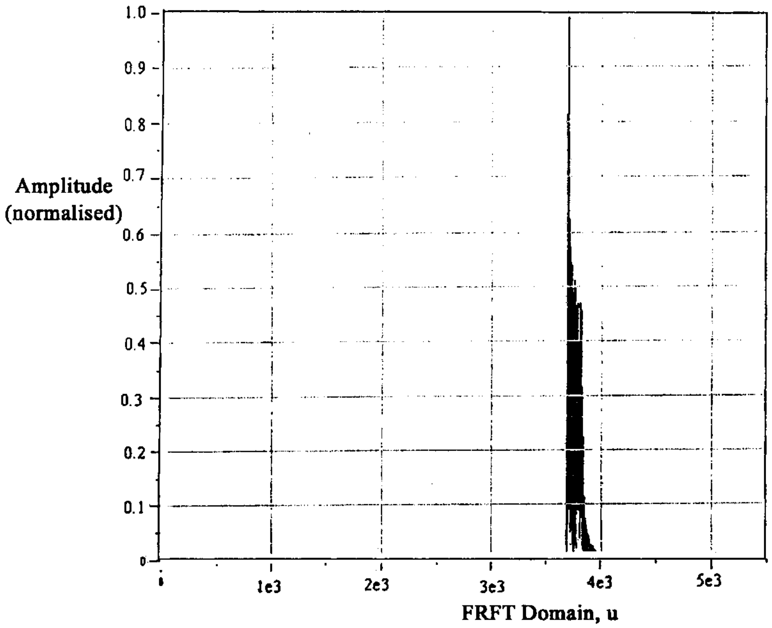


Fig 2 Fractional Fourier transform at optimum angle

rotated axis can be seen in Fig 5.

Our experimental results indicate that, the angle spoken of above, is indeed the angle at which the energy of the quadratic chirp is most concentrated. The idea of relative projection for various angles is graphically depicted in Fig 6.

In simulation too, we obtain the optimum fractional Fourier domain to be at an angle α,

$$\alpha = \pi / 2 + \tan^{-1}(3a(t_1 + t_2) + 2b)$$

where a, b are parameters of the chirp, $f(t) = e^{j(at^3 + bt^2 + ct)}$ and t_1, t_2 are as described above. In this manuscript, we aim at reporting this interesting result which also has an intuitive basis. We do not have a formal proof at the moment, but efforts in that direction are in progress.

3.1. Simulation Results

A quadratic chirp of the form $f(t) = e^{j(at^3 + bt^2 + ct)}$ is sampled at $t = nT$, with $T = 0.001s$. The other parameters of the chirp as follows, $a = 1/30$, $b = 4.85$, $c = 255$. The chirp is observed from $t_1 = 40s$ to $t_2 = 70s$. A plot of the

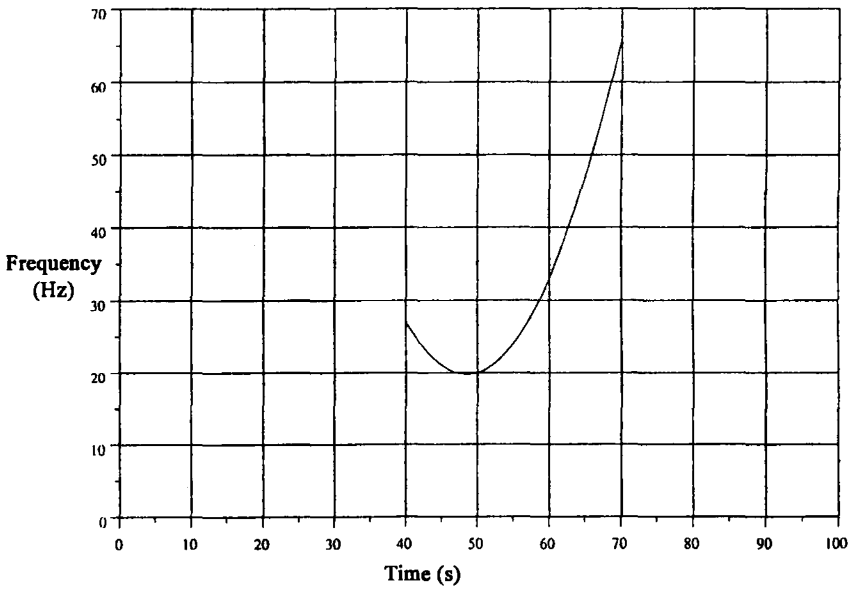


Fig 3 Quadratic chirp in time-frequency plane

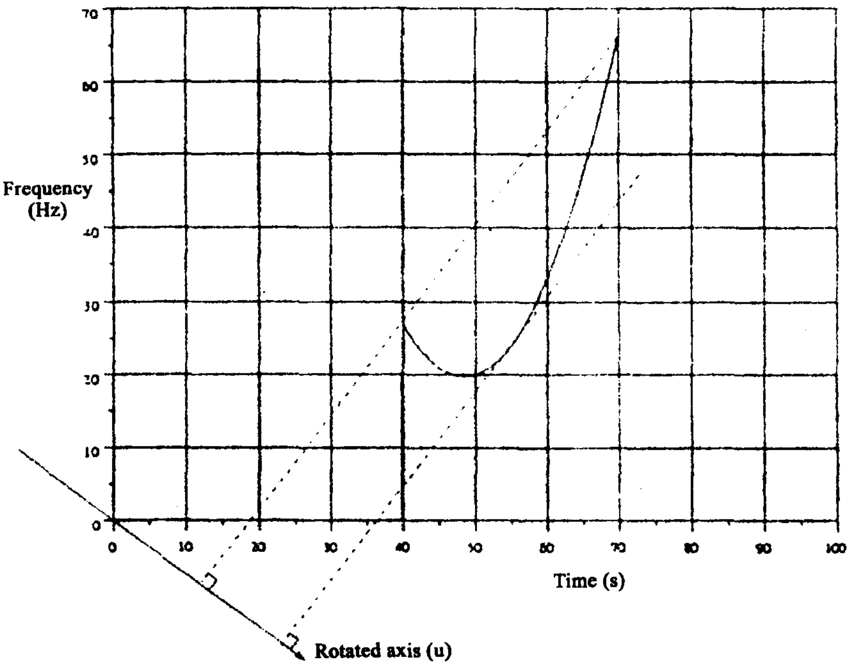


Fig 4 Quadratic chirp with rotated axis in time frequency plane

conventional Fourier transform is shown in Fig 7, while that of the fractional Fourier Transform at the optimum angle in Fig 2. From the plots it is observed that the FRFT at the optimum angle has a significantly lesser spread than that of the Fourier transform. Now, to suitably quantify the spread of a signal, we speak of its Radius of Gyration, R†. A plot of the experimentally observed radius of gyration versus angle (of the FRFT) is presented in Fig 8. It is observed that the radius of gyration passes through a minima (at 148°, equal to 4.367 units) close to the expected optimum angle (at 142°, equal to 4.618 units). From Fig 8 it is seen that the radius of gyration for the FRFT at 148° is 4.618 units while the radius of gyration is 13.41 units for the Fourier transform, substantially larger. The Fourier transform (Fig 2) and the fractional fourier transform (Fig 8) can be contrasted keeping in mind that the ratio of the maxima in the optimum fractional Fourier spectrum to the maxima in the Fourier spectrum is found to be 0.99.

†R of a signal $f(x)$ is defined as $R^2 = \int_{-\infty}^{+\infty} (x-x_0)^2 g(x)\,dx$. Here, $g(x) = \frac{|f(x)|^2}{\int_{-\infty}^{+\infty} |f(x)|^2 dx}$ and $x_0 = \int_{-\infty}^{+\infty} x g(x)\,dx$.

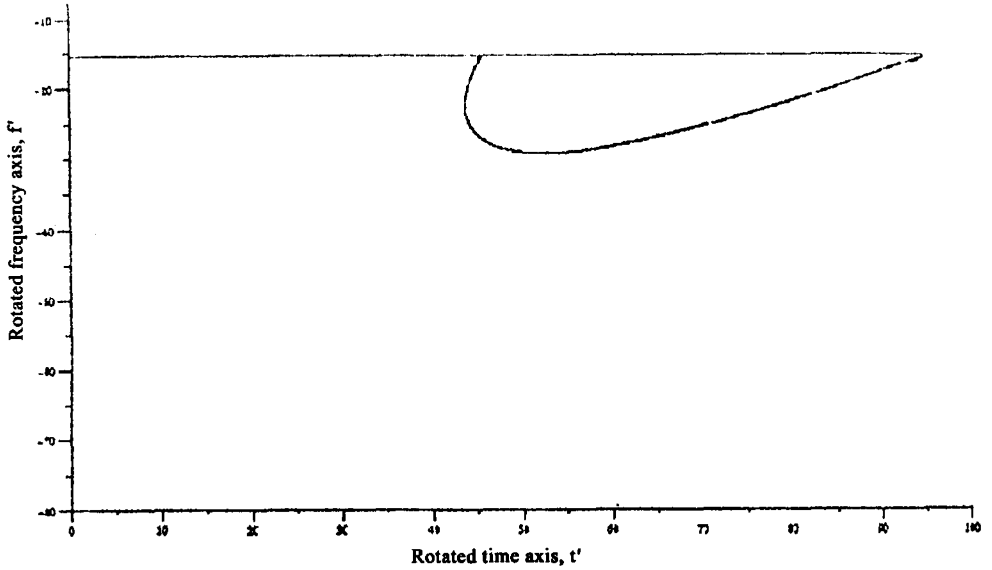


Fig 5 Related chirp in time frequency plane

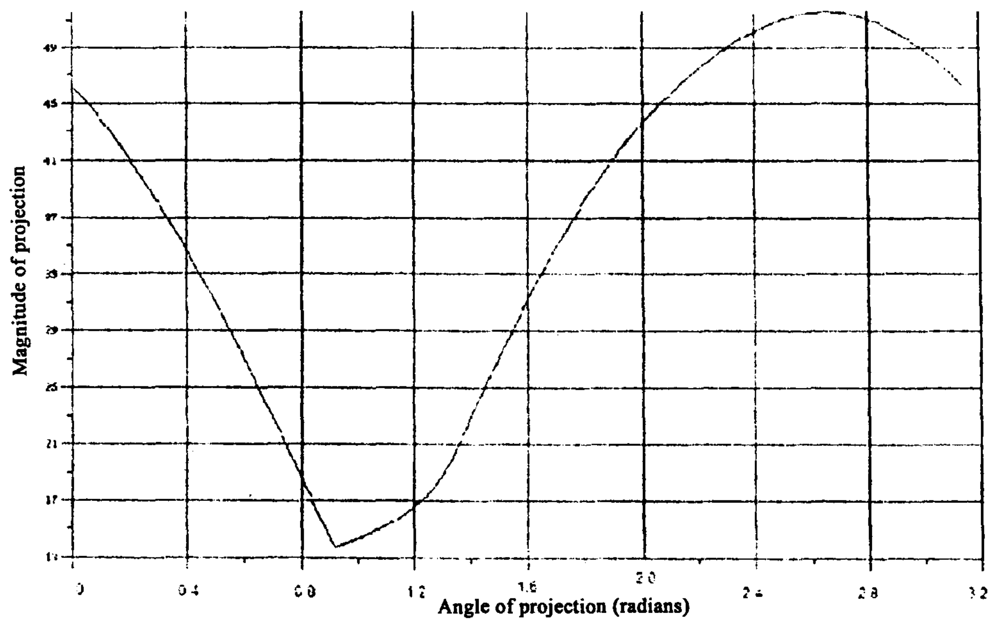


Fig 6 Projection for various angles

4. CONCLUSION

We have found an optimum fractional Fourier angle at which a quadratic chirp focuses reasonably well, that is, its energy is concentrated in a smaller region than the conventional Fourier transform (the FRFT at $\frac{\pi}{2}$). A comparison of Fig 6 (Projection of the Quadratic chirp on rotated axis as a function of the angle of rotation) and Fig 8 (Radius of gyration as a function of the angle of the FRFT) confirms the intuition that the FRFT can be thought of as a projection of a signal in the time-frequency plane onto a rotated axis.

Note that, the experimental plot of the radius of gyration (Fig 8) is observed to be $\frac{\pi}{2}$ radians ahead of the theoretical plot of the projection of the Quadratic chirp onto a rotated axis (Fig 6), as expected.

A simple analytical result for the optimum angle is obtained; the angle is found to be α,

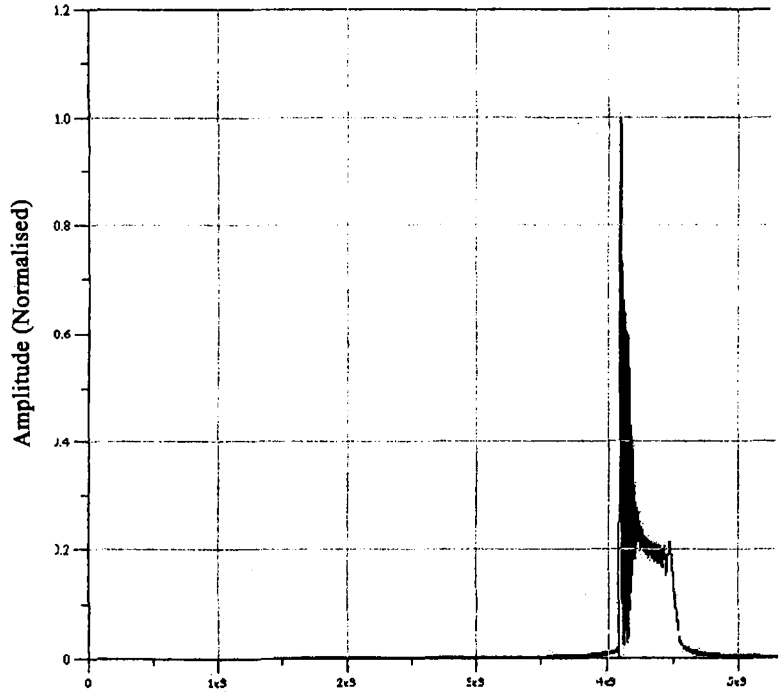


Fig 7 Fourier transform

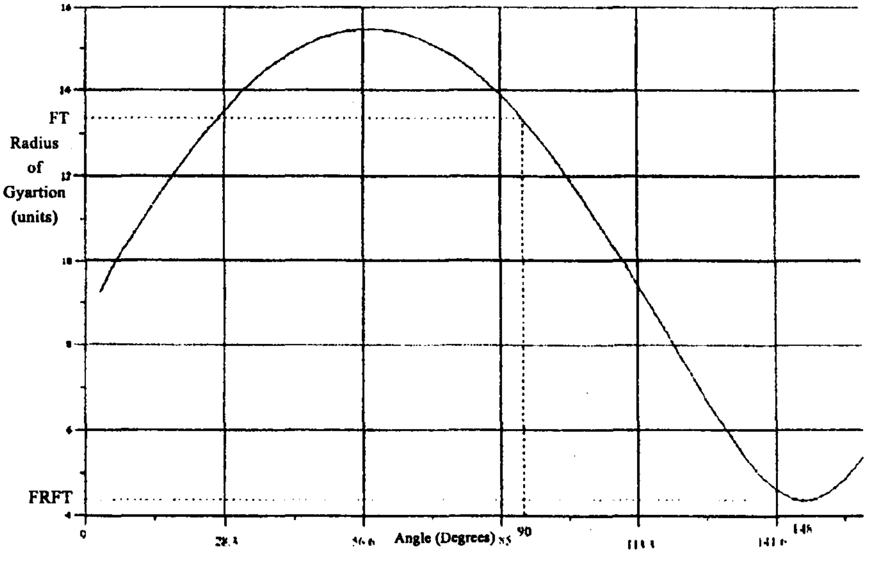


Fig 8 Radius of gyration for $0 < \alpha < \pi$

$$\alpha = \pi / 2 + \tan^{-1} (3a (t_1 + t_2) + 2b)$$

where a,b are parameters of the chirp which is observed for a duration from time t_1 to t_2.

Appendix - *A*

Proof for linear chirps:

We explain here, how to find the angle α, at which a linear chirp, given by $f(t) = e^{j(bt^2 + ct)}$ concentrates to an impulse at an appropriate fractional Fourier angle. Let

$$F_\alpha(u) = m\delta(u - u_0),$$

where m is the strength of the impulse, and u_0 is the location of the impulse on the v axis. From (2),

$$f(t) = \int_{-\infty}^{+\infty} \delta(u - u_0) K_{-\alpha}(u,t)\, du.$$

Thus,

$$f(t) = \left\{ \sqrt{\frac{1 - j\cot(\alpha)}{2\pi}}\, m e^{-j\frac{u_0^2 - \cot\alpha)}{2}} \right\} e^{j\left(-\frac{t^2\cos(\alpha)}{2} + \frac{tu_0}{\sin(\alpha)}\right)}$$

Comparing one gets

$$b = -\frac{\cot(\alpha)}{2}, c = \frac{u_0}{\sin(\alpha)}$$

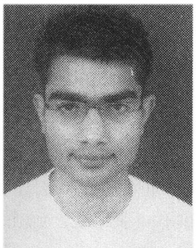
$$\Rightarrow \alpha = \tan^{-1}\left(\frac{-1}{2b}\right) = \frac{\pi}{2} + \tan^{-1}(2b)$$

Thus, a domain does indeed exist in which the linear chirp focusses to an impulse.

REFERENCES

1. L B Almeida, The Fractional Fourier Transform and Time-Frequency Representations, *IEEE Trans Signal Processing,* vol 42, no 11, pp 3084-3091, Nov 1994.
2. X G Xia, On bandlimited signals with fractional fourier transform, *IEEE Signal Processing Lett,* vol 3, no 3, pp 72-74, Mar 1996.
3. L B Almeida, An introduction to the angular fourier transform, in *Proc IEEE Conf, Acoustics, Speech, Signal Processing,* Minneapolis, MN, Apr 1993.
4. G S L Hong-Bo sun & H G W M Su, Application of the fractional fourier transform to moving target detection in airborne synthetic aperture radar, *IEEE Trans Aerosp Electron Syst,* vol 38, no 4, Oct 2002.

AUTHORS



Uday Khankhoje has completed his BTech degree (2001-05) from the Department of Electrical Engineering, Indian Institute of Technology (IIT) Bombay. He is now a PhD student at Caltech, Pasadena, USA. He has had an excellent performance during his Undergraduate (BTech) programme, and won several honours and accolades during his tenure as a student. The work reported in this paper, is a result of research carried out by him in the Undergraduate Research Opportunities Programme (UROP-01), in collaboration with Prof Vikram M Gadre, the co-author.

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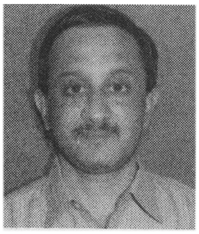

Vikram M Gadre is a member of the faculty of the Department of Electrical Engineering, IIT Bombay. He has served as Research Advisor to Uday Khankhoje in the UROP-01 Research Effort, of which this paper is the result.

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