EE7500: Advanced Electromagnetics — Homework Assignment 2

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Released: 25 Mar, due: 14 Apr 2026 Total: 100 points

Typed solutions (e.g. LaTeX) are highly preferred to handwritten & scanned submissions.

Academic Integrity:

- This homework must be completed and submitted individually.
- If you discuss the general approach with classmates, acknowledge them by name in your submission (no penalty for doing so).
- If you use AI tools (ChatGPT, Claude, etc.), clearly state which problems, how you used them, and the specific prompts. *If you use machine learning tools as a black box without building your own understanding, it is only the machines that will learn—not you.*
- Your PDF must begin with: "I hereby certify that this submission was generated by me with the highest academic integrity — *your name*"
- Submit: PDF with all solutions/plots, and MATLAB .m files for computational problems as one consolidated zip file. MATLAB files must be commented and named `questionXX.m`; the PDF filename must be your roll number.

1. **Small Current Loop and the Duality Theorem**

   A small circular loop of electric current I_0 with radius $a \ll \lambda$ lies in the xy-plane, centered at the origin. Its magnetic dipole moment is $m = I_0 \pi a^2$ (SI units: $\mathrm{A\,m^2}$).

   (a) (6 points) Starting from the vector potential integral

   $$\mathbf{A}(\mathbf{r}) = \frac{\mu I_0 a}{4\pi} \int_0^{2\pi} \hat{\phi}' \, \frac{e^{-jkR}}{R} \, d\phi', \qquad R = |\mathbf{r} - \mathbf{r}'|,$$

   use the far-field approximation $R \approx r - a \sin\theta \cos(\phi - \phi')$ for amplitude and phase, and the small-loop condition $ka \ll 1$, to show that

   $$\mathbf{A} = \frac{j\mu k m \sin\theta}{4\pi r} \, e^{-jkr} \, \hat{\phi}.$$

   From this, derive the far-field H_θ and E_ϕ.

   Hint: After expanding $e^{jka\sin\theta\cos(\phi-\phi')} \approx 1 + jka \sin\theta \cos(\phi - \phi')$, the zeroth-order integral vanishes and you will need $\int_0^{2\pi} \hat{\phi}' \cos(\phi - \phi') \, d\phi' = \pi \hat{\phi}$, where $\hat{\phi}' = (-\sin\phi', \cos\phi', 0)$ and $\hat{\phi} = (-\sin\phi, \cos\phi, 0)$.

   (b) (6 points) Recall the far-field of a Hertz electric dipole (current I_0, length ℓ):

   $$E_\theta = j\eta \, \frac{k I_0 \ell \sin\theta}{4\pi r} \, e^{-jkr}, \qquad H_\phi = \frac{E_\theta}{\eta}.$$

   Apply the duality substitution ($\mathbf{E}_A \to \mathbf{H}_F$, $\mathbf{H}_A \to -\mathbf{E}_F$, $\eta \to 1/\eta$, $I_0\ell \to I_m\ell$) to obtain H_θ and E_ϕ for a magnetic dipole $\mathbf{M} = I_m \ell \, \hat{z}$. By comparing with the loop results of part (a), find the equivalence between the small loop and the magnetic dipole currents, and also verify dimensional consistency.

2. **Uniqueness Theorem — The Critical Role of Loss**

   (a) (4 points) Suppose two solutions $\{\mathbf{E}_1, \mathbf{H}_1\}$ and $\{\mathbf{E}_2, \mathbf{H}_2\}$ satisfy Maxwell's equations in a bounded region V (bounded by surface S) with the same sources $\mathbf{J}, \mathbf{M}$ and the same tangential $\mathbf{E}$ prescribed on S. Define difference fields $\delta\mathbf{E} = \mathbf{E}_1 - \mathbf{E}_2$, $\delta\mathbf{H} = \mathbf{H}_1 - \mathbf{H}_2$. Write down the complex Poynting theorem for the difference fields integrated over V, and use the boundary condition to show that

$$\oint_S (\delta\mathbf{E} \times \delta\mathbf{H}^*) \cdot d\mathbf{S} = 0.$$

Then take the real part of the volume integral form of the Poynting theorem and state what it implies.

(b) (4 points) Identify the *one step* in the proof above that breaks down for a perfectly lossless medium (i.e. $\varepsilon'' = 0$, $\mu'' = 0$, $\sigma = 0$). Describe, with a brief physical example, a situation where fields in a lossless region are genuinely non-unique.

3. **LTE (4G) Base Station Coverage**

A 4G LTE base station (eNodeB) operates at $f = 1.8\,\text{GHz}$ (LTE Band 3) with the following parameters obtained from standards and datasheets:

- Transmit power: $P_T = 20\,\text{W}$ (43 dBm) [3GPP TS 36.104]
- Base station antenna gain: $G_T = 18\,\text{dBi}$ [typical sector panel antenna]
- Mobile handset antenna gain: $G_R = 0\,\text{dBi}$ (isotropic reference)
- Radiation efficiencies: $\eta_T = 0.90$ (base station), $\eta_R = 0.75$ (handset with body effects)
- Minimum required received power: $P_{\min} = -100\,\text{dBm}$ [3GPP TS 36.101, §7.3]

(a) (6 points) Using the Friis transmission equation $P_R = P_T G_T G_R \eta_T \eta_R \left(\frac{\lambda}{4\pi R}\right)^2$, compute the maximum coverage radius R (in km) under free-space propagation. Show all steps clearly.

(b) (4 points) Compute the corresponding coverage area (in km^2). Real-world LTE macro cells have radii of 1–50 km depending on terrain. By what approximate factor does your free-space result overestimate the practical coverage radius, and what is the primary physical reason? To answer this, assume a real-world cell radius to be 10 km. *Hint*: In part (a) we saw that the path loss factor was $\propto R^{-2}$. In real world scenarios this is not so, and the empirical factor is of the type R^{-n} where $n > 2$. Do some background reading to understand why $n \neq 2$ and estimate it in this problem.

4. **Antenna RCS Minimization without the Resonance Assumption**

The scattered field from an antenna with load Z_L is given by (Balanis 2-127 and 2-129):

$$\mathbf{E}^s(Z_L) = \mathbf{E}^s(0) - \frac{I_s}{I_t}\frac{Z_L}{Z_L + Z_A}\mathbf{E}^t \qquad \text{(2-127)}$$

$$\mathbf{E}^s(Z_L) = \mathbf{E}^s(0) - \frac{I_s}{I_t}\frac{1 + \Gamma_A}{2}\mathbf{E}^t, \quad \Gamma_A = \frac{Z_L - Z_A}{Z_L + Z_A} \qquad \text{(2-129)}$$

where $Z_A = R_A + jX_A$ is the antenna impedance, $\mathbf{E}^s(0)$ is the scattered field under short-circuit load, and $\mathbf{E}^t$ is the transmit-mode far field with terminal current I_t. Also use Balanis (2-131): $I_s Z_A = 2R_A I_m^*$, where I_m^* is the terminal current at conjugate match $Z_L = Z_A^*$.

(a) (6 points) Without assuming $X_A = 0$, substitute $Z_L = Z_A^*$ into (2-129) to find $\mathbf{E}^s(Z_A^*)$. Subtract this from the general (2-129) to arrive at a decomposition of the form

$$\mathbf{E}^s(Z_L) = \underbrace{\mathbf{E}^s(Z_A^*)}_{\text{structural}} - \underbrace{\frac{I_m^*}{I_t}\Gamma_c\,\mathbf{E}^t}_{\text{antenna mode}},$$

where $\Gamma_c \equiv (Z_L - Z_A^*)/(Z_L + Z_A)$. Show all algebra explicitly.

Note: Balanis uses $\Gamma^* = (Z_L - Z_A^*)/(Z_L + Z_A^*)$; explain under what condition $\Gamma_c = \Gamma^*$.

(b) (6 points) The RCS (from the magnitude of 2-129) in any observation direction (θ_0, ϕ_0) can be written as

$$\sigma = \left| \sqrt{\sigma_s} - (1 + \Gamma_A)\sqrt{\sigma_a}\, e^{j\phi_r} \right|^2 ,$$

where σ_s, σ_a are the structural and antenna-mode RCS, and ϕ_r is their relative phase. For a fixed antenna geometry (fixed Z_A, σ_s, σ_a, ϕ_r), derive the optimal load Z_L that sets $\sigma(\theta_0, \phi_0) = 0$, and express it in terms of $\alpha \equiv \sqrt{\sigma_s/\sigma_a}$ and ϕ_r. State the condition on α for the solution to require an active (non-passive) load.

5. **Hertz Dipole Radiation Pattern — MATLAB**

The radiation intensity of a Hertz dipole is $U(\theta) = U_0 \sin^2\theta$ (W/sr), independent of ϕ.

Reference: `https://www.mathworks.com/help/matlab/ref/polarplot.html`

(a) (4 points) **Pattern visualization.** Write a MATLAB script that plots $U(\theta)/U_0$ in a 2D polar plot as a function of $\theta \in [0°, 360°]$, with labeled axes and a title.

(b) (6 points) **Directivity calculation.** Write MATLAB code to numerically compute the directivity:

$$D_{\max} = \frac{4\pi\, U_{\max}}{P_{\rm rad}}, \qquad P_{\rm rad} = \int_0^{\pi}\int_0^{2\pi} U(\theta)\sin\theta\, d\theta\, d\phi.$$

Use a fine angular grid (e.g. 10^4 points in θ, uniform over $[0, \pi]$). Verify numerically that $D_{\max} = 3/2$ (confirm to at least 3 significant figures), and identify the direction(s) of maximum radiation.

Deliverables: MATLAB .m file with comments, the 2D polar plot with labeled axes, the numerical value of $D_{\max}$ printed to the command window, and a 2–3 sentence summary of what the plot reveals about the Hertz dipole radiation pattern.

6. **Quarter-Wave Monopole and Image Theory**

(a) (5 points) A quarter-wave ($\lambda/4$) vertical monopole antenna is mounted on an infinite PEC ground plane. Using image theory, identify the image source, describe the equivalent problem valid for $z > 0$, and write down the far-field E_θ of the monopole for $0 \le \theta \le \pi/2$. You may use the result for the far-field of a center-fed half-wave dipole in free space (total length $\lambda/2$, current I_0) from existing sources without derivation.

(b) (5 points) Given that the radiation resistance of a half-wave dipole is $R_{\rm rad}^{\rm dipole} = 73\,\Omega$, find the radiation resistance $R_{\rm rad}^{\rm mono}$ of the quarter-wave monopole. Provide a careful argument based on radiated power and the symmetry of the radiation pattern. *Hint:* Go to the fundamental definition of radiation resistance.

7. **Effective Aperture and the Friis Formula**

(a) (4 points) Starting from the Friis transmission equation $P_r = P_t G_t G_r (\lambda/4\pi R)^2$ (Balanis 2-116, lossless matched case), derive the relation $A_e = G\lambda^2/(4\pi)$ (Balanis 2-71) by identifying the effective aperture as $A_e \equiv P_r/S_{\rm inc}$, where $S_{\rm inc}$ is the power density incident on the receiving antenna. State clearly which antenna's gain and aperture appear in your expression.

(b) (4 points) A half-wave dipole antenna ($D_{\max} = 1.64$, $R_{\rm rad} = 73\,\Omega$, radiation efficiency $\eta_{\rm rad} = 100\%$) is used as a receive antenna at $f = 500\,$MHz. A plane wave with power density $S_{\rm inc} = 10\,\mu$W/m^2 is incident from the direction of maximum gain, with matching polarization and a matched load. Compute: (i) the maximum effective aperture A_e in cm^2; (ii) the power delivered to the matched load.

8. **Two-Element Phased Array — MATLAB**

Two isotropic point sources are placed along the x-axis, separated by $d = \lambda/2$. The sources carry currents $I_1 = I$ and $I_2 = Ie^{j\delta}$. The array factor is:

$$\mathrm{AF}(\phi) = 1 + e^{j(\pi\cos\phi + \delta)},$$

where ϕ is the azimuth angle measured from the $+x$ axis.

Reference: `https://www.mathworks.com/help/matlab/ref/polarplot.html`

(a) (6 points) Write a MATLAB script to compute and plot $|\mathrm{AF}(\phi)|$ vs. $\phi \in [0°, 360°]$ as polar plots on a single figure for three cases: $\delta = 0$ (broadside), $\delta = -\pi/2$ (beam steered toward $\phi = 60°$), and $\delta = -\pi$ (endfire toward $\phi = 0°$). Label each curve. For each case, state the angle(s) of maximum $|\mathrm{AF}|$.

(b) (4 points) For the broadside case ($\delta = 0$), derive the half-power beamwidth (HPBW) analytically. Verify your result numerically by finding the two half-power angles in MATLAB and confirming they match within 1°.

Deliverables: MATLAB .m file with comments, the polar plot with all three curves labeled, and the numerically determined HPBW printed to the command window.

9. **Near-Field vs. Far-Field Transition — MATLAB**

The H_ϕ field of a Hertz dipole (in the $\theta = 90°$ plane, $\sin\theta = 1$) is:

$$H_\phi = \frac{kI_0\ell}{4\pi r} \underbrace{j}_{\text{far-field term}} + \underbrace{\frac{I_0\ell}{4\pi r^2}}_{\text{near-field term}},$$

with far-field amplitude $|H_{\mathrm{FF}}| = kI_0\ell/(4\pi r)$ and near-field amplitude $|H_{\mathrm{NF}}| = I_0\ell/(4\pi r^2)$.

(a) (5 points) Write a MATLAB script that plots $|H_{\mathrm{FF}}|$ and $|H_{\mathrm{NF}}|$ (normalized to $I_0\ell/(4\pi)$, so the two plotted quantities are $k/r = kr/r^2$ and $1/r^2$) on a single log-log plot as a function of kr over the range $kr \in [0.01, 100]$. Include axis labels, a legend, and a vertical line at the transition point.

(b) (5 points) Determine analytically the value of kr at which $|H_{\mathrm{FF}}| = |H_{\mathrm{NF}}|$ (the near-field/far-field transition). At $f = 1\,\mathrm{GHz}$, what is the corresponding physical distance r? Express this distance both in meters and in wavelengths λ.

Deliverables: MATLAB .m file, the log-log plot with labeled axes and vertical transition line, and a 2–3 sentence physical interpretation of the transition printed as a comment.

10. **Friis Radar Range Equation**

A monostatic radar operates at $f = 10\,\mathrm{GHz}$ (X-band) with the following parameters: peak transmit power $P_T = 50\,\mathrm{kW}$, transmit/receive antenna gain $G_T = G_R = 30\,\mathrm{dBi}$, and minimum detectable received power $P_{\min} = -90\,\mathrm{dBm}$. The radar range equation is:

$$P_R = \frac{P_T G_T G_R \lambda^2 \sigma}{(4\pi)^3 R^4},$$

where σ is the target's radar cross-section (RCS).

(a) (5 points) Compute the maximum detection range R (in km) for a target with $\sigma = 1\,\mathrm{m}^2$ (approximately 0 dBsm, roughly a human body). Show all steps.

(b) (5 points) Repeat for a stealth aircraft with $\sigma = 10^{-3}\,\mathrm{m}^2$ ($-30\,\mathrm{dBsm}$). By what factor does the maximum detection range decrease compared to part (a)? Express this factor also in dB.