EE7500: Advanced Electromagnetics — Homework Assignment 1

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Released: 12 Feb, due: 23 Feb 2026 Total: 100 points

Academic Integrity:

- This homework must be completed and submitted individually.
- If you discuss the general approach with classmates, acknowledge them by name in your submission (no penalty for doing so).
- If you use AI tools (ChatGPT, Claude, etc.), clearly state which problems, how you used them, and the specific prompts. *If you use machine learning tools as a black box without building your own understanding, it is only the machines that will learn—not you.*
- Submit: PDF with all solutions/plots, and MATLAB .m files for computational problems as one consolidated zip file. The matlab files should have comments and named like so: 'questionXX.m' and the PDF file name must be your roll number.
- Your main PDF should start with a self-certification of academic integrity like so: "I hereby certify that this document and associated code has been generated by me with the highest academic integrity – your name"

1. **Transmission Line Analysis**

   A lossless transmission line with $Z_0 = 50\,\Omega$ at $f = 1\,\text{GHz}$ is terminated with $Z_L = 75 + j50\,\Omega$. The line has length $\ell = 0.15\lambda$.

   (a) (3 points) Calculate the load reflection coefficient Γ_L in both rectangular and polar forms, and the VSWR.

   (b) (3 points) Find the input impedance Z_{in}.

   (c) (2 points) If $P_{inc} = 10\,\text{W}$, find the power delivered to the load and the power reflected.

   (d) (2 points) Sketch the voltage magnitude $|V(d)|$ along the line as a function of distance d from the load, indicating the locations of the first maximum and first minimum.

2. **Quarter-Wave Transformer Design**

   Design a quarter-wave matching network: $Z_L = 100\,\Omega$ to $Z_0 = 50\,\Omega$ at $f_0 = 2.4\,\text{GHz}$ on a substrate with $\varepsilon_r = 4.4$ (FR-4).

   (a) (2 points) Calculate the characteristic impedance Z_{QW} of the quarter-wave section.

   (b) (2 points) Calculate the physical length of the matching section in millimeters.

   (c) (2 points) Estimate the fractional bandwidth over which $|\Gamma_{in}| < 0.1$.

   (d) (2 points) Draw a labeled schematic showing the source line (Z_0), the quarter-wave matching section (indicating its impedance and physical length), and the load (Z_L).

3. **Skin Depth and Shielding Effectiveness**

   A two-layer planar shield is illuminated by a normally-incident plane wave at $f = 200\,\text{MHz}$. Layer 1 (outer): aluminum, $t_1 = 100\,\mu\text{m}$, $\sigma_{Al} = 3.5 \times 10^7$ S/m. Layer 2 (inner): copper, $t_2 = 50\,\mu\text{m}$, $\sigma_{Cu} = 5.8 \times 10^7$ S/m. Both have $\mu_r = 1$; negligible air gap.

   (a) (3 points) Calculate the skin depth $\delta_s = 1/\sqrt{\pi f \mu_0 \sigma}$ for Al and Cu at 200 MHz. Express each layer thickness in units of skin depth, then compute the absorption shielding effectiveness SE $\approx 8.686\,(t/\delta_s)$ dB for each layer and the total SE.

(b) (3 points) Compare with a single 150 μm aluminum layer (same total thickness). Which configuration provides better shielding? Explain physically why.

(c) (2 points) At $f = 5\,\text{GHz}$, a plane wave with surface magnetic field $H_0 = 1\,\text{A/m}$ is incident on a thick copper slab. Using the impedance boundary condition $Z_s = (1+j)/(\sigma\delta_s)$, compute $\vec{E}_{tan}$ at the surface.

(d) (2 points) The absorption-only SE formula used above ignores re-reflections at layer boundaries. Under what condition on t/δ_s is this approximation justified? Is it valid for the shield in this problem?

Ignore reflection losses at boundaries for the SE calculations.

4. **Polarization and Wave Superposition**

A plane wave at $f = 10\,\text{GHz}$ propagates in the $+z$ direction in free space. Its phasor electric field is $\vec{E} = \left(a_x\,\hat{x} + a_y\,e^{j\delta}\,\hat{y}\right)e^{-jk_0 z}$ V/m, with $e^{j\omega t}$ time convention.

(a) (3 points) For $a_x = 3$, $a_y = 4$, $\delta = 0$: identify the polarization type. Find the tilt angle ψ of the electric field vector relative to the x-axis and compute the time-averaged power density $|\vec{S}_{av}|$.

(b) (3 points) For $a_x = 3$, $a_y = 4$, $\delta = -\pi/2$: write the time-domain electric field $\vec{E}(z=0,t)$ and evaluate it at $t = 0$, $T/4$, $T/2$, $3T/4$. Sketch the locus of the $\vec{E}$-field tip in the x-y plane. Identify the polarization type, the sense of rotation (right-hand or left-hand), and compute the axial ratio.

(c) (3 points) Two co-polarized plane waves of the same frequency arrive at a receiver:

$$\vec{E}_1 = 2\,\hat{x}\,e^{-jk_0 z}\ \text{V/m (direct path)},$$
$$\vec{E}_2 = 3\,\hat{x}\,e^{-j(k_0 z+\pi/3)}\ \text{V/m (reflected path with additional phase } \pi/3).$$

Compute the time-averaged Poynting vector magnitude for each wave individually ($|\vec{S}_1|$, $|\vec{S}_2|$) and for the total field $\vec{E} = \vec{E}_1 + \vec{E}_2$. Verify that $|\vec{S}_{total}| \neq |\vec{S}_1| + |\vec{S}_2|$ and explain physically why superposition does not apply to the Poynting vector.

(d) (3 points) The reflected wave now has a different polarization due to ground reflection:

$$\vec{E}_1 = 2\,\hat{x}\,e^{-jk_0 z}\ \text{V/m (horizontally polarized, direct path)},$$
$$\vec{E}_2 = 1.5\,\hat{y}\,e^{-j(k_0 z+\pi/4)}\ \text{V/m (vertically polarized, reflected path)}.$$

What is the polarization of the total field $\vec{E}_1 + \vec{E}_2$? Is it linear, circular, or elliptical? Determine the sense of rotation.

5. **Vector Potential from a Current Distribution**

A line current along the z-axis: $\vec{J}(\vec{r}\,') = I_0\,e^{-|z'|/a}\,\delta(x')\delta(y')\,\hat{z}$, where I_0 has units of A and a is a length.

(a) (3 points) Using the 3D free-space Green's function $G(\vec{r},\vec{r}\,') = \frac{e^{-jk|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}$, write the integral for the vector potential $\vec{A}(\vec{r}) = \mu\int_{V'}\vec{J}(\vec{r}\,')G(\vec{r},\vec{r}\,')\,dV'$.

(b) (2 points) Simplify by performing the x' and y' integrations. State the remaining integral and its limits.

(c) (3 points) Compute $\nabla'\cdot\vec{J}$ and show it is *not* zero. Using the continuity equation $\nabla\cdot\vec{J} = -j\omega\rho$, what does this imply about the charge density and scalar potential ϕ?

(d) (2 points) Explain physically why $\vec{A}$ must point in the $\hat{z}$ direction everywhere for this source.

You do not need to evaluate the integral in (b).

6. **Verification of 1D Green's Function**

   The 1D Green's function $g(x, x') = \frac{-j}{2k_0} e^{-jk_0|x-x'|}$ satisfies $\frac{d^2 g}{dx^2} + k_0^2 g = -\delta(x - x')$.

   (a) (3 points) For $x > x'$, write g without the absolute value. Compute dg/dx and d^2g/dx^2, then substitute into $d^2g/dx^2 + k_0^2 g$ and show it equals zero.

   (b) (2 points) Repeat the above for $x < x'$.

   (c) (2 points) Compute the jump in dg/dx at $x = x'$: $\left.\frac{dg}{dx}\right|_{x'+} - \left.\frac{dg}{dx}\right|_{x'-}$. Show it equals -1.

   (d) (3 points) Integrate the defining equation over $[x' - \epsilon, x' + \epsilon]$ and take $\epsilon \to 0$. Show that the result is consistent with part (c).

7. **1D Green's Function — MATLAB Symbolic Toolbox**

   Use MATLAB's Symbolic Math Toolbox to verify properties of the 1D Green's function.

   Reference: `https://www.mathworks.com/help/symbolic/`

   (a) (2 points) **Symbolic definition.** Define $g_1(x)$ for $x > x'$ and $g_2(x)$ for $x < x'$ as symbolic expressions (with x, x', k_0 as symbolic variables).

   (b) (3 points) **Symbolic verification of Helmholtz equation.** For each region: compute dg/dx, d^2g/dx^2, then $d^2g/dx^2 + k_0^2 g$. Use `simplify` to verify the result is zero. Display expressions at each step.

   (c) (2 points) **Derivative discontinuity.** Symbolically evaluate dg_1/dx at $x = x'$ (from right) and dg_2/dx at $x = x'$ (from left). Compute the jump and verify it equals -1.

   (d) (3 points) **Numerical plotting.** Convert to numeric using `matlabFunction`. For $k_0 = 2\pi$, $x' = 0$, plot $\text{Re}\{g\}$ and $\text{Im}\{g\}$ as two subplots over $x \in [-3, 3]$. Verify $g(1, 0) = g(0, 1)$ numerically (symmetry).

   Deliverables: MATLAB .m file with comments, intermediate symbolic outputs printed to the command window, plots with labeled axes, and a 2–3 sentence summary of what the symbolic verification confirms about the Green's function.

8. **Sommerfeld Radiation Boundary Condition**

   The general homogeneous solution for the 2D Green's function is $g(r) = a\, H_0^{(1)}(kr) + b\, H_0^{(2)}(kr)$.

   The Sommerfeld radiation boundary condition (for $e^{j\omega t}$ convention) is: $\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial g}{\partial r} + jkg\right) = 0$.

   Asymptotic forms: $H_0^{(1)}(kr) \sim \sqrt{\frac{2}{\pi kr}}\, e^{j(kr-\pi/4)}$; $H_0^{(2)}(kr) \sim \sqrt{\frac{2}{\pi kr}}\, e^{-j(kr-\pi/4)}$.

   (a) (2 points) For $H_0^{(2)}(kr)$: compute $\frac{\partial}{\partial r} H_0^{(2)}(kr)$ using the asymptotic form (leading order), substitute into the radiation condition, and verify it is satisfied.

   (b) (2 points) Repeat for $H_0^{(1)}(kr)$ and show it does *not* satisfy the condition.

   (c) (4 points) The DLMF (`https://dlmf.nist.gov/10.17`) gives higher-order asymptotic expansions for the Hankel functions (which equations are relevant?). Write out the *two-term* expansion of $H_0^{(2)}(kr)$ and show that it also satisfies the Sommerfeld radiation condition.

   (d) (2 points) Why are Hankel functions preferred over Bessel functions J_0, Y_0 for radiation problems?

   Hint for (a)–(b): Use the product rule; keep only the leading-order term in $1/r$.

9. **Dimensional Comparison of Green's Functions**

The free-space Green's functions are:

$$\text{1D: } g_{1D} = \frac{-j}{2k} e^{-jk|x-x'|}, \quad \text{2D: } g_{2D} = \frac{-j}{4} H_0^{(2)}(k|\vec{r} - \vec{r}'|), \quad \text{3D: } g_{3D} = \frac{e^{-jk|\vec{r}-\vec{r}'|}}{4\pi|\vec{r} - \vec{r}'|}$$

In the far field, $|g_{2D}| \sim 1/\sqrt{2\pi kr}$ for $r \gg \lambda$.

(a) (2 points) Extract the amplitude decay factor $|g|$ vs. distance for each case and state the decay law (constant, $1/\sqrt{r}$, $1/r$).

(b) (3 points) Using energy conservation, derive the decay rates from first principles:

- 1D: power flows along one direction only.
- 2D: power spreads over a cylindrical surface of circumference $2\pi r$.
- 3D: power spreads over a sphere of area $4\pi r^2$.

Since field amplitude $\propto \sqrt{\text{intensity}}$, verify consistency with part (a).

(c) (3 points) A finite line source of length L radiates in 3D space. In the near field ($r \ll L$) it looks like an infinite line source; in the far field ($r \gg L$) it looks like a point source. Which Green's function describes each regime? Estimate the transition distance.

(d) (2 points) Explain in 2–3 sentences:

- Why the 1D Green's function has no amplitude decay.
- Why all real (finite-extent) antennas eventually exhibit 3D $1/r$ decay in the far field.

10. **1D Green's Function in a Layered Medium**

Consider the 1D Helmholtz equation with a piecewise-constant wavenumber modeling an interface between two non-magnetic ($\mu_r = 1$) dielectric half-spaces:

$$\frac{d^2 g}{dx^2} + k^2(x)\, g = -\delta(x - x'), \qquad k(x) = \begin{cases} k_1 = \omega\sqrt{\mu_0 \varepsilon_0 \varepsilon_{r1}}, & x < 0 \\ k_2 = \omega\sqrt{\mu_0 \varepsilon_0 \varepsilon_{r2}}, & x > 0 \end{cases}$$

A point source is located at $x = x' > 0$ (inside medium 2). Both g and dg/dx are continuous at the interface $x = 0$ (TE polarization, same μ).

(a) (3 points) Write the general solution for $g(x)$ in each of the three regions: $x < 0$, $0 < x < x'$, and $x > x'$. Apply the outgoing-wave (radiation) condition at $x \to \pm\infty$ to discard the appropriate terms. How many unknown coefficients remain?

(b) (3 points) Write the four boundary conditions (two at $x = 0$, two at $x = x'$) and solve for the coefficients. Express your answer in terms of k_1, k_2, x', and the reflection coefficient $\Gamma = (k_2 - k_1)/(k_1 + k_2)$.

(c) (2 points) Verify that when $\varepsilon_{r1} = \varepsilon_{r2}$ (homogeneous medium, $\Gamma = 0$), your solution reduces to the free-space Green's function $g = \frac{-j}{2k} e^{-jk|x-x'|}$.

(d) (2 points) For $\varepsilon_{r1} = 1$ and $\varepsilon_{r2} = 4$: compute Γ and describe qualitatively how $|g(x)|$ differs from the homogeneous case. In which region does a partial standing wave appear, and why? What happens to the amplitude transmitted into medium 1?