Floquet Theory: Fourier Series, Floquet Series, and Scattering by Periodic Structures

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Abstract

This document provides a concise overview of Floquet theory, beginning with standard Fourier series before deriving the Floquet series for quasi-periodic functions. These mathematical principles are then applied to analyze transverse electric (TE) plane wave scattering by a periodic dielectric structure. Finally, the text formulates the reflected and transmitted fields as a superposition of Floquet modes and derives the grating equation governing propagating and evanescent waves.

I. Fourier Series

For a function $f(t)$ that is strictly periodic with period T,

$$f(t+T) = f(t),$$

it admits the Fourier series expansion:

$$f(t) = \sum_{m=-\infty}^{\infty} a_m \, e^{j\frac{2\pi m}{T}t} \tag{1}$$

with coefficients

$$a_m = \frac{1}{T}\int_0^T f(t)\, e^{-j\frac{2\pi m}{T}t}\, dt.$$

The spectral lines sit at the discrete frequencies $\omega_m = 2\pi m/T$, which are integer multiples of the fundamental $\omega_0 = 2\pi/T$.

II. Floquet Series

A. Setup: Quasi-Periodic Functions

Now consider a function $f(t)$ satisfying the Floquet (quasi-periodicity) condition:

$$f(t+T) = e^{jp}\, f(t), \qquad p \in \mathbb{R}.$$

The amplitude $|f(t)|$ is periodic with period T, but the phase advances by a fixed amount p each period. A standard Fourier series is not directly applicable because the overall period of $f(t)$ may be much larger than T (or infinite). However, a Fourier-like expansion — the Floquet series — can be derived as follows.

B. Construction from a Base Function

We build $f(t)$ as a superposition of time-shifted copies of a base function $f_0(t)$, with a progressive phase taper:

$$f(t) = \sum_{n=-\infty}^{\infty} f_0(t-nT)\, e^{jnp}. \tag{2}$$

Verification of the quasi-periodicity condition:

$$f(t+T) = \sum_n f_0\big(t+T-nT\big)\, e^{jnp} = \sum_n f_0\big(t-(n-1)T\big)\, e^{jnp}.$$

Substituting $m = n-1$ (relabelling the dummy index):

$$= \sum_m f_0(t-mT)\, e^{j(m+1)p} = e^{jp}\sum_m f_0(t-mT)\, e^{jmp} = e^{jp}\, f(t).$$

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C. Fourier Transform of $f(t)$

Using the convention $\tilde{F}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$, substituting the superposition and exchanging sum and integral:

$$\tilde{F}(\omega) = \frac{1}{2\pi}\sum_n e^{jnp}\int_{-\infty}^{\infty} f_0(t-nT)\, e^{-j\omega t}\, dt.$$

Applying the shift theorem with $t' = t - nT$:

$$\int_{-\infty}^{\infty} f_0(t-nT)\, e^{-j\omega t}\, dt = e^{-j\omega nT}\int_{-\infty}^{\infty} f_0(t')\, e^{-j\omega t'}\, dt' = 2\pi\, \tilde{F}_0(\omega)\, e^{-j\omega nT},$$

so

$$\tilde{F}(\omega) = \tilde{F}_0(\omega) \qquad \underbrace{\sum_{n=-\infty}^{\infty} e^{jn(p-\omega T)}}_{\text{to be evaluated by Poisson summation}} \qquad .$$

D. Poisson Summation Identity

The following identity (which can be proven via the Poisson summation formula, or by recognising the Fourier series of the Dirac comb) is used:

$$\sum_{n=-\infty}^{\infty} e^{jn\alpha} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\alpha - 2\pi m). \tag{3}$$

Setting $\alpha = p - \omega T$ and using the scaling property $\delta(aX) = \frac{1}{|a|}\delta(X)$:

$$\sum_n e^{jn(p-\omega T)} = 2\pi \sum_m \delta(p - \omega T - 2\pi m) = \frac{2\pi}{T}\sum_m \delta\left(\omega - \frac{p-2\pi m}{T}\right).$$

Re-labelling the dummy index $m \to -m$ (valid since m runs over all integers):

$$= \frac{2\pi}{T}\sum_m \delta(\omega - \omega_m), \qquad \omega_m \equiv \frac{2\pi m + p}{T} = \frac{2\pi m}{T} + \frac{p}{T}. \tag{4}$$

Therefore:

$$\tilde{F}(\omega) = \frac{2\pi}{T}\sum_{m=-\infty}^{\infty} \tilde{F}_0(\omega_m)\, \delta(\omega - \omega_m). \tag{5}$$

The spectrum of $f(t)$ is discrete, with spectral lines at the Floquet frequencies $\{\omega_m\}$ — these are the ordinary Fourier-series frequencies $2\pi m/T$, each uniformly shifted by p/T.

E. The Floquet Series

Applying the inverse Fourier transform $f(t) = \int_{-\infty}^{\infty} \tilde{F}(\omega)\, e^{j\omega t}\, d\omega$ and picking out the delta functions in the spectrum:

$$f(t) = \sum_{m=-\infty}^{\infty} a_m \exp\left(j\, \frac{2\pi m + p}{T}\, t\right) \tag{6}$$

where the Floquet coefficients are $a_m = \dfrac{2\pi}{T}\, \tilde{F}_0(\omega_m)$.

	Condition	Spectral lines	Series form
Fourier	$f(t+T)=f(t)$	$\omega_m = 2\pi m/T$	$\sum_m a_m\, e^{j(2\pi m/T)t}$
Floquet	$f(t+T)=e^{jp}f(t)$	$\omega_m = (2\pi m + p)/T$	$\sum_m a_m\, e^{j(2\pi m+p)t/T}$

F. Comparison with the Fourier Series and Factored Form

Setting $p=0$ in the Floquet series immediately recovers the standard Fourier series. Alternatively, the Floquet function can be written in the factored form:

$$f(t) = e^{jpt/T} \underbrace{\sum_m a_m\, e^{j\frac{2\pi m}{T}t}}_{=:\, g(t),\ \text{strictly periodic with period } T},$$

confirming that a Floquet function is the product of a linear phase ramp $e^{jpt/T}$ and a periodic function $g(t)$.

Block rule (dummy variable): Since t in the series is a dummy variable, the result holds for any variable. For a function $f(x)$ satisfying $f(x+d) = e^{jp}f(x)$, replace $t \to x$ and $T \to d$:

$$f(x) = \sum_{m=-\infty}^{\infty} a_m \exp\!\left(j\,\frac{2\pi m + p}{d}\,x\right). \tag{7}$$

This block is invoked directly whenever the Floquet condition is established.

III. Plane Wave Scattering by a Periodic Dielectric Structure

A. Problem Geometry and Assumptions

Geometry:

- Region I ($z>0$): free space ($\epsilon_r = 1$). Contains the incident and reflected fields.
- Interface at $z=0$: front face of the periodic structure.
- Region II ($z<0$): periodic medium, $\epsilon_r(x+d) = \epsilon_r(x)$, with ϵ_r a function of x only.
- Transmission region (z large negative): free space below the structure, contains the transmitted field.

Assumptions:

1) 2D problem: No y-dependence ($\partial/\partial y = 0$); wave vector confined to the x-z plane.
2) TE polarisation: $\vec{E} = E_y(x,z)\,\hat{y}$.
3) Non-magnetic medium: $\mu = \mu_0$ everywhere.
4) Time convention: $e^{j\omega t}$, suppressed throughout.

Incident wave (propagating in the $-z$ direction from Region I, at angle θ_i to the z-axis):

$$E_y^{inc}(x,z) = E_0\, e^{-j(k_x x - k_{z0} z)}, \qquad k_x = k_0 \sin\theta_i, \quad k_{z0} = k_0\cos\theta_i, \tag{8}$$

with $k_0 = \omega/c = \omega\sqrt{\mu_0\epsilon_0}$ and $k_x^2 + k_{z0}^2 = k_0^2$.

B. From Maxwell's Equations to the Wave Equation

Starting from the time-harmonic Maxwell equations:

$$\nabla \times \vec{E} = -j\omega\mu_0\vec{H}, \qquad \nabla\times\vec{H} = j\omega\epsilon_0\epsilon_r(x)\,\vec{E}.$$

With $\vec{E} = E_y\hat{y}$ and $\partial_y = 0$, the first equation gives:

$$\nabla\times(E_y\hat{y}) = -\frac{\partial E_y}{\partial z}\hat{x} + \frac{\partial E_y}{\partial x}\hat{z} = -j\omega\mu_0\vec{H},$$

so the non-zero magnetic field components are:

$$H_x = \frac{1}{j\omega\mu_0}\frac{\partial E_y}{\partial z}, \qquad H_z = -\frac{1}{j\omega\mu_0}\frac{\partial E_y}{\partial x}, \qquad H_y = 0. \tag{9}$$

For the second Maxwell equation, with $H_y = 0$ and $\partial_y = 0$, only the $\hat{y}$-component survives:

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\epsilon_0\epsilon_r(x)\, E_y. \tag{10}$$

Substituting the field components into this equation and using $k_0^2 = \omega^2\mu_0\epsilon_0$:

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} + k_0^2\,\epsilon_r(x)\,E_y = 0. \tag{11}$$

In Region I ($\epsilon_r = 1$), this reduces to the standard Helmholtz equation $\nabla^2 E_y + k_0^2 E_y = 0$.

C. Establishing the Floquet Condition

The incident wave picks up a phase factor under a shift $x \to x + d$:

$$E_y^{inc}(x + d,\, z) = e^{-jk_x d}\, E_y^{inc}(x, z).$$

Since $\epsilon_r(x + d) = \epsilon_r(x)$, shifting the entire problem by d in x leaves the scattering structure unchanged. By the uniqueness theorem, the total field must undergo the same phase shift:

$$E_y^{total}(x + d,\, z) = e^{-jk_x d}\, E_y^{total}(x, z). \tag{12}$$

This is precisely the Floquet condition $f(x + d) = e^{jp} f(x)$ with $p = -k_x d$.

D. Floquet Series Expansion of the Fields

Applying the block rule with $x \to x$, $d \to d$, and $p = -k_x d$:

$$E_y(x, z) = \sum_{m=-\infty}^{\infty} \mathcal{E}_m(z)\, e^{-jk_{xm}x}, \tag{13}$$

where the Floquet (tangential) wavenumbers are, after re-labelling the dummy index $m \to -m$:

$$k_{xm} = k_x + \frac{2\pi m}{d} = k_0 \sin\theta_i + \frac{2\pi m}{d}, \qquad m \in \mathbb{Z}. \tag{14}$$

The $m = 0$ mode carries the tangential wavenumber of the incident wave. Each successive order is shifted by the grating's reciprocal lattice vector $2\pi/d$.

E. Decoupled Modes in the Free-Space Regions

Substituting the expansion into the free-space Helmholtz equation in Region I:

$$\sum_m \left[\frac{d^2\mathcal{E}_m}{dz^2} + \left(k_0^2 - k_{xm}^2\right)\mathcal{E}_m\right] e^{-jk_{xm}x} = 0.$$

Since the set $\{e^{-jk_{xm}x}\}$ consists of functions with distinct wavenumbers, they are linearly independent and each bracket must vanish independently:

$$\frac{d^2\mathcal{E}_m}{dz^2} + k_{zm}^2\,\mathcal{E}_m = 0, \qquad k_{zm}^2 = k_0^2 - k_{xm}^2. \tag{15}$$

This is a decoupled 1D Helmholtz equation for each Floquet mode m.
Branch convention for k_{zm}:

$$k_{zm} = \sqrt{k_0^2 - k_{xm}^2}, \quad \mathrm{Re}(k_{zm}) \geq 0, \quad \mathrm{Im}(k_{zm}) \leq 0.$$

For evanescent modes ($k_{xm}^2 > k_0^2$) this gives $k_{zm} = -j\kappa_m$ with $\kappa_m = \sqrt{k_{xm}^2 - k_0^2} > 0$. With this branch:

- Reflected form $e^{-jk_{zm}z} = e^{-\kappa_m z}$: decays as $z \to +\infty$ in Region I.
- Transmitted form $e^{+jk_{zm}z} = e^{+\kappa_m z}$: for $z < 0$ this equals $e^{-\kappa_m |z|}$, decaying as $z \to -\infty$ in the transmission region.

F. Reflected and Transmitted Fields

In Region I, only the $m = 0$ Floquet mode carries the incident wave (propagating in $-z$); all other modes have no incoming component. The incident amplitude is E_0 for $m = 0$ and zero otherwise, while R_m is the amplitude of the m-th reflected/diffracted mode (propagating in $+z$):

$$E_y^{ref}(x,z) = \sum_{m=-\infty}^{\infty} R_m \, e^{-jk_{xm}x - jk_{zm}z}. \tag{16}$$

The total field in Region I (incident + reflected) is:

$$E_y^{I}(x,z) = \underbrace{E_0 \, e^{-j(k_x x - k_{z0} z)}}_{\text{incident}} + \sum_{m=-\infty}^{\infty} R_m \, e^{-j(k_{xm}x + k_{zm}z)}. \tag{17}$$

In the transmission region (z large negative, below the structure), only modes propagating in the $-z$ direction are present:

$$E_y^{trans}(x,z) = \sum_{m=-\infty}^{\infty} T_m \, e^{-jk_{xm}x + jk_{zm}z}. \tag{18}$$

Both fields are Floquet series in x, each term being a plane wave (or evanescent wave) at one of the discrete tangential wavenumbers $\{k_{xm}\}$.
The amplitudes R_m and T_m are determined by matching the tangential fields (E_y and H_x) across the interface at $z = 0$, which in general requires solving the coupled-mode equations inside the periodic region.

G. Propagating vs. Evanescent Modes and the Grating Equation

For each order m:

$$k_{zm} = \sqrt{k_0^2 - k_{xm}^2} = \sqrt{k_0^2 - \left(k_0 \sin\theta_i + \frac{2\pi m}{d}\right)^2}. \tag{19}$$

- Propagating ($k_{xm}^2 < k_0^2$, k_{zm} real and positive): mode m carries power to/from the far field.
- Evanescent ($k_{xm}^2 > k_0^2$, $k_{zm} = -j\kappa_m$): mode m decays exponentially away from the structure and carries no far-field power.

To find the coefficients R_m and T_m requires matching boundary conditions at $z = 0$ (continuity of E_y and $H_x = \frac{1}{j\omega\mu_0}\partial_z E_y$), which couples the free-space Floquet modes to the field inside the periodic structure.