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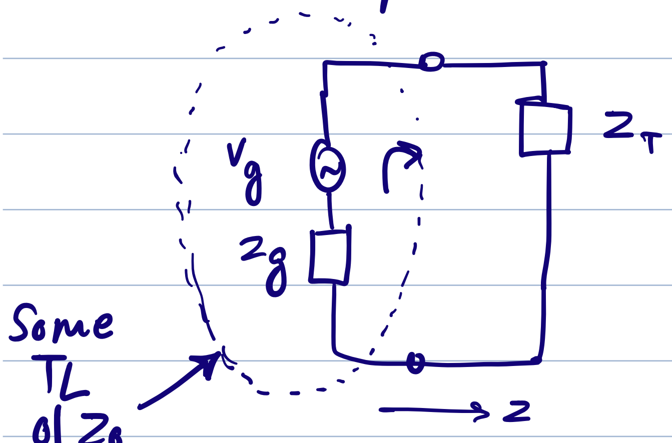
# Microwave Network Theory

Lumped circuit equivalent of Tx ant:




, Reflection coefficient:

$$\Gamma = \frac{Z_T - Z_0}{Z_T + Z_0}$$

Near the port $V = V_0^+ e^{-j\beta z} + V_0^- e^{-j\beta z}$

and $\frac{V_0^-}{V_0^+} = \Gamma.$

Rewrite as:

$$\therefore \quad V_0^- = \Gamma \; V_0^+$$

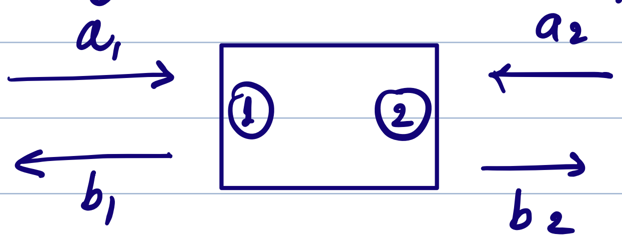
$V_0^+$ ← incident ↓ given the symbol 'a'

$V_0^-$ reflected ↓ given the symbol 'b'

This is a simple 1-port network $b = S\,a$

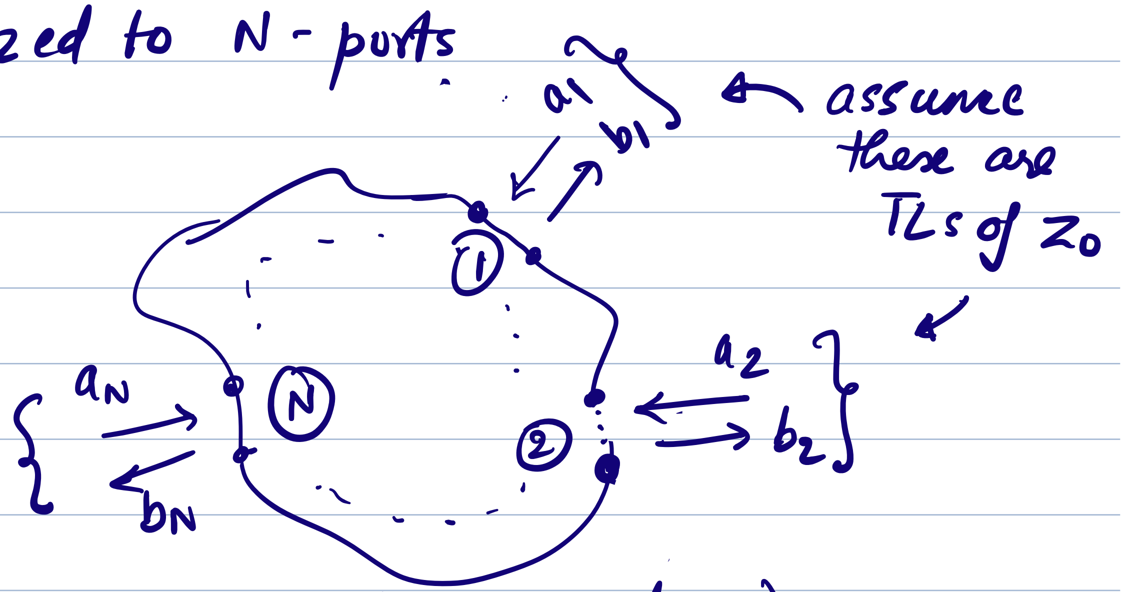
$S$ is called the scattering matrix, also called S-parameters.

↳ This can be generalized to 2 parameters




$$\therefore \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

or generalized to N-ports




Similarly giving $\begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix} = S \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}$

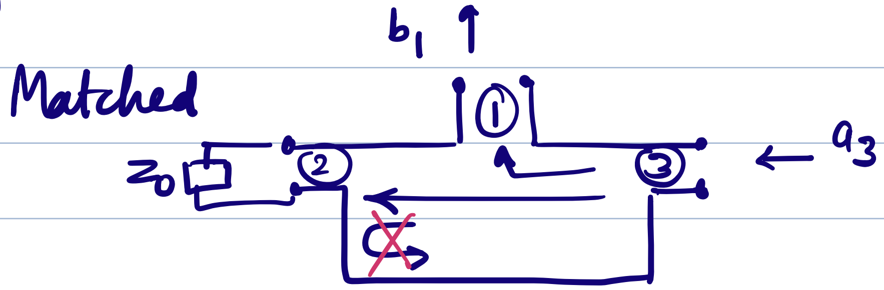
So $S_{ij} = \left. \dfrac{b_i}{a_j} \right|_{a_k = 0,\, k \neq j}$

Means what? ① Excite port $j$ <u>ONLY</u> with $a_j$

② Measure $b_i$ out of port $i$.

③ Make sure other ports have $a_k = 0$, i.e. no power from other ports leaks in. How to do this? Put matched loads at these ports.

eg. I want to measure $S_{13}$ (3 port network)




$S_{13} = \dfrac{b_1}{a_3}$.

3 properties of S-params w/o proof.

① For a reciprocal network (no nonlinear, active devices etc, same conds as the Lorentz reciprocity thm), $S = S^T$

② For a lossless network $S$ is a unitary matrix, i.e. $SS^H = I$ (identity).

$\Rightarrow \sum_j S_{ij} S^*_{kj} = \delta_{ik}$.

③ Recall that a, b are inc & reflected voltages e.g $a_n = V_n^+$, $b_n = V_n^-$. So the port voltages will be $V_n = V_n^+ + V_n^-$ ($z=0$ is taken to be at the port). Similarly the port current will be $I_n = I_n^+ - I_n^-$

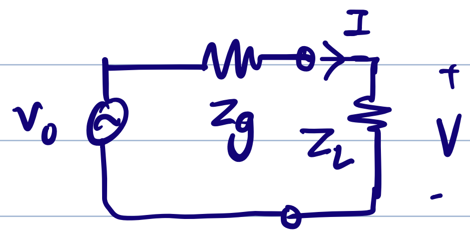
$$I_n = \frac{1}{Z_0}\left(V_n^+ - V_n^-\right) = \frac{a_n - b_n}{Z_0}$$

Zmatrix

$V_n = a_n + b_n$. Other matrices: $V = ZI$, $I = YZ$

— x —

The above discussion holds true when $Z_0$ is lossless and there is a TL connected at each port. e.g.




Here there is no voltage wave or TL (i.e. no $Z_0$). So, a generalization is: power waves. They can be applied to the above case as well as to regular TL cases.

To get the intuition consider a regular port with voltage V & current I & a TL of $Z_0$.

$$V = V_0^+ + V_0^- , \quad I = (V_0^+ - V_0^-)/Z_0$$

$$\Rightarrow \text{Power}, \; P_L = \frac{1}{2} Re(VI^*) = \frac{1}{2Z_0} Re\Big[ |V_0^+|^2 - |V_0^-|^2 \underbrace{- V_0^+ V_0^{-*} + V_0^{+*} V_0^-}_{\text{purely imag.}} \Big]$$

$$P_L = \frac{1}{2Z_0}\left[ |V_0^+|^2 - |V_0^-|^2 \right]$$

Now switch to the ckt above and let's try to define power wave amplitudes like so

$$a = \frac{V + Z_R I}{2\sqrt{R_R}} , \quad b = \frac{V - Z_R^* I}{2\sqrt{R_R}}$$

where we define $Z_R = R_R + jX_R$ as a reference impedance.

Now, calculate V, I in terms of a, b?

$$V = \frac{Z_R a + Z_R^* b}{\sqrt{R_R}} , \quad I = \frac{a - b}{\sqrt{R_R}}.$$

$$\Rightarrow P_L = \frac{1}{2} Re(VI^*) = \frac{1}{2}\left(|a|^2 - |b|^2\right)$$

Good, similar to our result above when $Z_0 = 1$.

Last bit, what about reln between a & b?

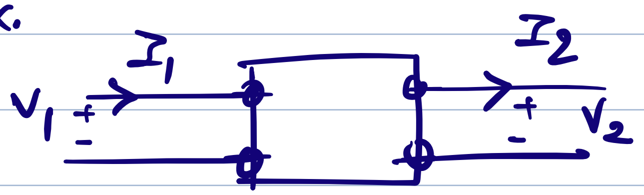
Use $V = Z_L I$ for the port

$$\frac{b}{a} = \Gamma_p = \frac{V - Z_R^* I}{V + Z_R I} = \frac{Z_L - Z_R^*}{Z_L + Z_R^*}.$$

Same as our usual $\Gamma$ when $Z_R$ is real.
Since $Z_R$ is arbitrary & in our hand, common choices are (i) $Z_R = Z_L^*$ or
(ii) $Z_R = Z_g$
chosen out of convenience depending on appl.

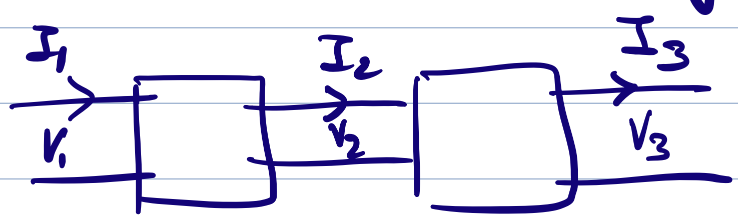
We can generalize this power wave defn for n-ports, choosing ref impedances for each port & get a matrix: $b = S_p a$.

<Aside: another useful matrix is the ABCD matrix.




$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

This is great for cascading!



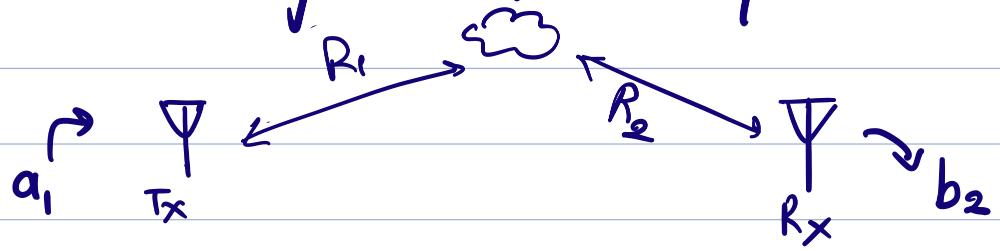

Thus

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} V_3 \\ I_3 \end{bmatrix}$$ & so on.

keep in mind the change in sign convention for $I_2$, $I_3$.

end aside>

Lets connect the S-parameters with EM fields in the case of 2 antenna systems.




$$\left[ a_n = \frac{V_n + Z_0 I_n}{2\sqrt{R_0}}, \quad b = \frac{V_n - Z_0^* I_n}{2\sqrt{R_0}} \qquad \text{Recal} \right]$$

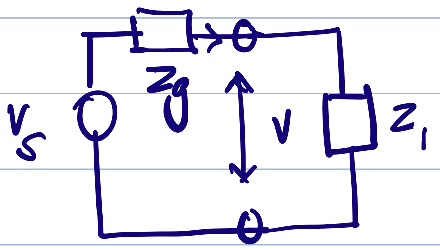
Logical flow: $\quad a_1 \longrightarrow obj \longrightarrow b_2, \quad S_{21} = \dfrac{b_2}{a_1}$

In more detail:

$$a_1 \xrightarrow[\vec{l}_1,\ R_1,\ Z_1, Z_0]{prop} \vec{E}_{inc}\Big|_{obj} \xrightarrow[R_2]{S^{obj}} \vec{E}^{scat}\Big|_{Rx} \xrightarrow[\vec{l}_2,]{} V_{oc} \xrightarrow[Z_2, Z_0]{} b_2$$

due to object

① At the tx port:




Assume a ref imp $Z_0$ (need not be real, need not be a TL).

$$a_1 = \frac{V_1 + Z_0 I_1}{2\sqrt{R_0}} = \frac{Z_1 I_1 + Z_0 I_1}{2\sqrt{R_0}} = \frac{(Z_1 + Z_0)}{2\sqrt{R_0}} I_1 \quad \text{—①}$$

② Field emitted by the antenna, landing on the object (farfield approx).

$$\vec{E}^{inc}(R_1) = j\frac{\eta k}{4\pi R_1} e^{-jkR_1} I_1 \underbrace{\vec{l}_1}_{\text{effictive vector length of ant}}$$

e.g. $\sin\theta\,\hat{\theta}$ for Hertz dipole

Let's make it more general: Instead of just $\vec{l}_1$, lets write it as $\begin{bmatrix} l_1^{co} \\ l_2^{xp} \end{bmatrix}$ → co-pol, x-pol

e.g. antenna generates both $\hat{\theta}$ & $\hat{\phi}$.

So $$\vec{E}_{inc} = \begin{bmatrix} E_i^{co} \\ E_i^{xp} \end{bmatrix} = j\frac{\eta k}{4\pi R_1} e^{-jkR_1} I_1 \underbrace{\begin{pmatrix} l_1^{co} \\ l_1^{xp} \end{pmatrix}}_{2\times 1 \text{ vector}} \quad ②$$

③ How does the object react? We model it as a polarimetric scatterer:

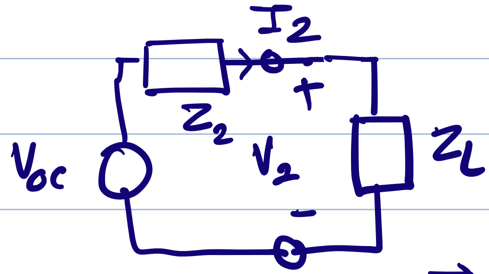
$$\vec{E}_{scat} = \frac{e^{-jkR_2}}{R_2} \underbrace{[S^{obj}]}_{2\times 2 \text{ matrix.}} \vec{E}_{inc}\Big|_{\text{at object}} \quad ③$$

We can put ② into ③.

$$\vec{E}_{scat} = \frac{j\eta}{4\pi} \frac{e^{-jk(R_1+R_2)}}{R_1 \cdot R_2} I_1 S^{obj} \begin{pmatrix} l_1^{co} \\ l_1^{xp} \end{pmatrix} \quad ④$$

④ Now we have to convert this $\vec{E}_{scat}$ into '$b_2$'.

$R_x$ ant ckt?




Thus $V_{oc} = \vec{E}_{scat} \cdot \vec{l}_2 = \vec{l}_2 \cdot \vec{E}_{scat}$

In our case: $V_{oc} = [\, l_2^{co} \;\; l_2^{xp} \,] \cdot \vec{E}_{scat}$ $\rightarrow \begin{bmatrix} E_s^{co} \\ E_s^{xp} \end{bmatrix}$

i.e. cross coupling can also contribute to $V_{oc}$.

$$\therefore V_{oc} = \frac{j\eta k}{4\pi} \frac{e^{-jk(R_1+R_2)}}{R_1 \;\; R_2} I_1 \underbrace{\underbrace{[\, l_2^{co} \;\; l_2^{xp} \,]}_{\vec{l}_2^{\,T}} S^{obj} \underbrace{\begin{bmatrix} l_1^{co} \\ l_2^{xp} \end{bmatrix}}_{\vec{l}_1}}_{\tau}$$

$\tau$ → purely fn of Tx / obj / Rx geometry.

from the ckt: $V_{oc} = I_2 Z_2 + V_2$ (KVL)

and $V_2 = Z_L I_2 \;\Rightarrow\; I_2 = V_{oc} / Z_2 + Z_L$

Recall the convention for $a_n, b_n$ had port v & I to be going <u>into</u> the port.

$$\Rightarrow a_2 = \frac{V_2 + Z_0(-I_2)}{2\sqrt{R_0}} = \frac{(Z_L - Z_0)\, I_2}{2\sqrt{R_0}}$$

[going into the port]

$$b_2 = \frac{V_2 - Z_0^*(-I_2)}{2\sqrt{R_0}} = \frac{(Z_L + Z_0^*)\, V_{oc}}{2\sqrt{R_0}\,(Z_2 + Z_L)}$$

(5) **Finally:**

$$T_{21} = \frac{b_2}{a_1} = \frac{(Z_L + Z_0^*)\, V_{oc}}{2\sqrt{R_0}\,(Z_2 + Z_L)} \times \frac{2\sqrt{R_0}}{(Z_0 + Z_1)\, I_1}$$

$$T_{21} = \frac{Z_L + Z_0^*}{(Z_2 + Z_L)(Z_0 + Z_1)} \frac{V_{oc}}{I_1} \quad \text{--- } (*)$$

$$T_{21} = \frac{(Z_L + Z_0^*)}{(Z_2 + Z_L)(Z_0 + Z_1)} \frac{j\eta k}{4\pi} \frac{e^{-jk(R_1 + R_2)}}{R_1 R_2} \, \tau$$

Subtle distinction betn $T_{21}$ & $S_{21}$.

$$S_{21} = b_2/a_1 \Big|_{a_2 = 0}.$$

$a_2 = 0 \Rightarrow Z_L = Z_0$ (matched load). Then,

$$S_{21} = \frac{j\eta k}{2\pi} \frac{e^{-jk(R_1 + R_2)}}{R_1 R_2} \frac{Re(Z_0)}{(Z_1 + Z_0)(Z_2 + Z_0)} \cdot \vec{l}_2^{\,T} S^{obj} \vec{l}_1$$

else

$$T_{21} = \frac{j\eta k}{4\pi} \frac{e^{-jk(R_1 + R_2)}}{R_1 R_2} \frac{(Z_L + Z_0^*)}{(Z_1 + Z_0)(Z_2 + Z_0)} \cdot l_2^T S^{obj} \vec{l}_1$$