Uday Khankhoje

Lorentz.

Reciprocity theorem for electromagnetics.

↳ Consider a linear, isotropic, heterogeneous medium with two sets of sources $\{J_i, M_i\}$ $i = 1, 2$, each producing fields $\{E_i, H_i\}$.

↳ The thm states that:

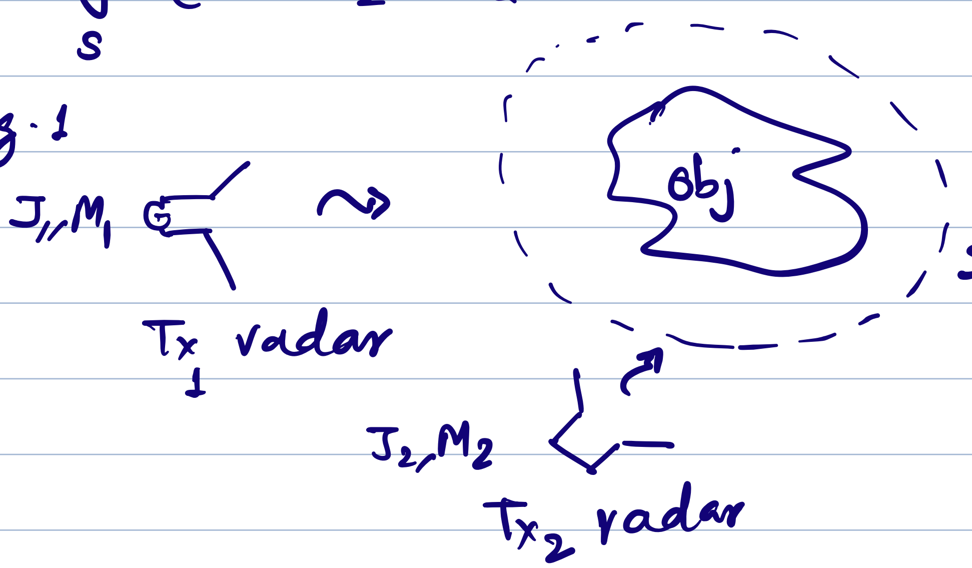
$$\oint_S (E_1 \times H_2 - E_2 \times H_1) \cdot ds = -\int_V (E_1 \cdot J_2 + H_2 \cdot M_1 - E_2 \cdot J_1 - H_1 \cdot M_2)\, dv$$

Some implications:

① Source free version:

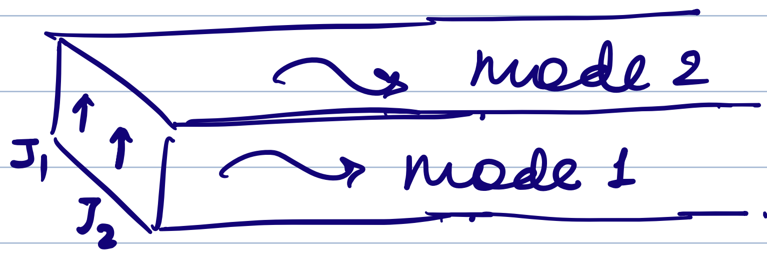
$$\oint_S (E_1 \times H_2 - E_2 \times H_1) \cdot ds = 0$$

e.g. 1



Then must hold.

e.g. 2: waveguide



Say that J_1 & J_2 excite a different modes in the waveguide, then also the fields must satisfy the above eqn over any closed surface excluding $\{J_i\}$.

(2) When we have sources but take $S \to \infty$ then by using the Sommerfeld radiation BC $\oint_S (\;) = 0$

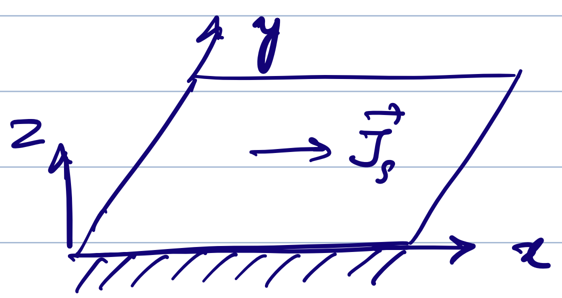
So: $$\underbrace{\int_V (E_1 \cdot J_2 - H_1 \cdot M_2)\, dv}_{} = \int_V (E_2 \cdot J_1 - H_2 \cdot M_1)\, dv$$

→ called the coupling or reaction of fields of source 1 to sources # 2.
→ often denoted in short as $\langle 1, 2 \rangle$.

$\therefore \quad \langle 1, 2 \rangle = \langle 2, 1 \rangle$

An e.g.

test dipole antenna



∞ PEC sheet.

$\therefore$ Source 1 : $\vec{J}_s$ sheet current impressed on PEC, $\bar{J}_1(r) = J_s\, \delta(z)\, \hat{x}$.

fields : E_1, H_1

Source 2: $\vec{J}_2 = J_d\, \delta(r - r')\, \hat{p}$ test dipole

fields E_2, H_2.

Since $M_1, M_2 = 0$, and $\delta \to \infty$, Lorentz R.T. $\Rightarrow$

$$\int_V E_1 \cdot J_2 \, dv = \int_V E_2 \cdot J_1 \, dv$$

$$J_d\, \vec{E}_1(r') \cdot \hat{p} = J_s \int_{xy\ \text{plane}} \vec{E}_2(r) \cdot \hat{x} \, dx\, dy$$

Notice the RHS. On the PEC, $E_{tan} = 0$
$\Rightarrow$ RHS $= 0$. Same would happen for any tangential direction of current on the surface.

$$\Rightarrow \quad \vec{E}_1(r') \cdot \hat{p} = 0 .$$

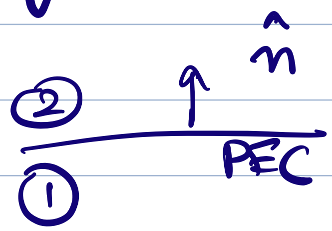
Since the dipole placement & orientation is arbitrary, this implies that

$\vec{E}_1(r') = 0 \Rightarrow$ an impressed J current on a PEC sheet does not radiate!

— x —

An imp question arises. What happened to the tangential-H boundary condn at the interface? i.e.

$$\hat{n} \times (\bar{H}_2 - \bar{H}_1) = \bar{J}$$



Inside the PEC there are no fields, so $H_1 = 0 \Rightarrow \hat{n} \times \bar{H}_2 = \bar{J}$.

But we just showed that the impressed $\bar{J}$ doesn't radiate!

$$\Rightarrow \bar{H}_2 = 0.$$

The resolution to this apparant paradox is that the $\bar{J}$ on the RHS is the total $\vec{J}$, not just the impressed $\bar{J}$.

$\therefore$ The PEC has an induced $\bar{J}_{ind}$ equal and opp to the impressed $\vec{J}_{imp}$ such that total $\vec{J} = \vec{J}_{ind} + \bar{J}_{imp} = 0$.

Now the B.C. reads $0 = 0$,

(which is fine).

↳ Quick proof of Reciprocity thm.

Key vector calculus identity:

$$\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$$

Now:

$$\nabla \times E_1 = -M_1 - j\omega\mu H_1 \quad — ①$$
$$\nabla \times H_1 = +J_1 + j\omega\varepsilon E_1 \quad — ②$$

$$\nabla \times E_2 = -M_2 - j\omega\mu H_2 \quad — ③$$
$$\nabla \times H_2 = J_2 + j\omega\varepsilon E_2 \quad — ④$$

$H_2 \cdot$ eq ① $-$ $E_1 \cdot$ eq ④

LHS: $H_2 \cdot \nabla \times E_1 - E_1 \cdot \nabla \times H_2$

$$= \nabla \cdot (E_1 \times H_2) \quad — ⑤$$

Similarly $H_1 \cdot$ eq ③ $-$ E_2 eq ②

LHS: $H_1 \cdot \nabla \times E_2 - E_2 \cdot \nabla \times H_1$

$$= \nabla \cdot (E_2 \times H_1) \quad — ⑥$$

Now we do ⑤ − ⑥, after algebra

$$\nabla \cdot (E_1 \times H_2 - E_2 \times H_1) = -\left[E_1 \cdot J_2 + H_2 \cdot M_1 - E_2 \cdot J_1 - H_1 \cdot M_2\right]$$

This is in differential form.

By applying divergence thm,

$$\oint_S (E_1 \times H_2 - E_2 \times H_1) \cdot ds = -\int_V \left[(E_1 \cdot J_2 + H_2 \cdot M_1) - (E_2 \cdot J_1 + H_1 \cdot M_2) \right] dv$$

This is in integral form. QED