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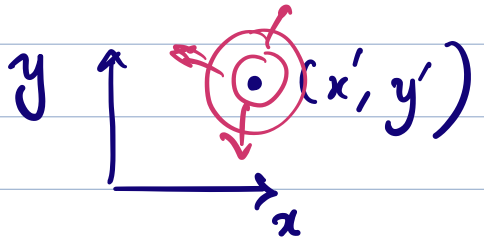
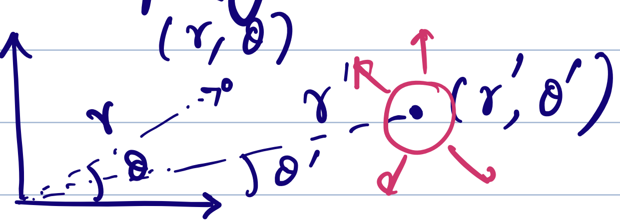
Moving now to the 2D Green's function.

Defn $$\nabla^2 g(\bar{r}, \bar{r}') + k_0^2 g(\bar{r}, \bar{r}') = -\delta(\bar{r} - \bar{r}')$$

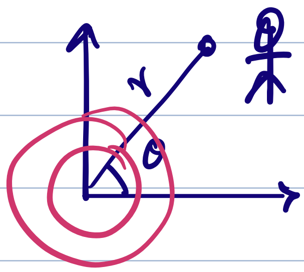
we can do (x, y) or (r, θ) coords.
Think $\rightarrow$ which would simplify math.



or



In both case we have a 2D propagation.
But what if we place $\vec{r}'$ at the origin?



Will this guy see any θ depn of the field at different θs?
No! Let's exploit this obsv.

$$\therefore \nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \quad \text{simplifies to:}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}$$

$\therefore$ Green's fn defn (mult by r^2) $\rightarrow$ $$r^2 \frac{d^2}{dr^2} g + r\frac{d}{dr} g + k_0^2 r^2 g = -r^2 \delta(r)$$

(1)

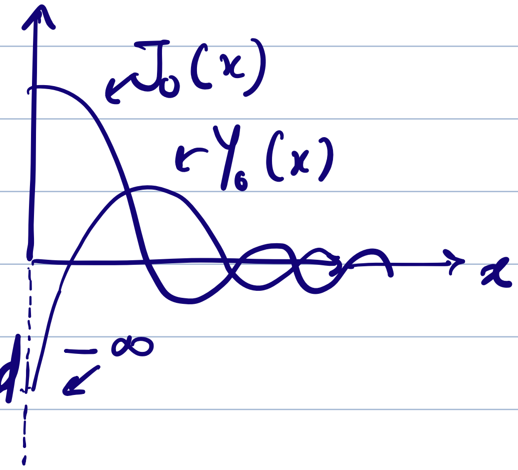
where $g(r, r'=0) \rightarrow g(r)$.
(short hand)

↳ Turns out this is similar to a very well known diff eqn: Bessels diff eq. In standard form:

$$x^2 y'' + x y' + (x^2 - \underbrace{\alpha^2}_{\text{const.}}) y = 0. \quad \text{— ②}$$

2nd order D.E ⇒ 2 indepn solns

ⓐ either $\underbrace{J_\alpha(x)}_{\text{first kind.}}$, $Y_\alpha(x)$ → second kind.



or a linear combination

ⓑ of them $H_\alpha^{(1)}(x)$, $H_\alpha^{(2)}(x)$

Hankel fn: first & second kind

We can choose our soln set as either ⓐ or ⓑ.
keep RHS in mind … Y looks "better" than J.
⇒ Set ⓑ might be more appropriate.

↳ How do we pour ① into ② ? Ignore RHS for now.
key is the $k_0^2 r^2 g \Leftrightarrow x^2 y$ term.

$$\Rightarrow \text{subs} \quad k_0 r = x \quad \Rightarrow \quad k_0 r' = x'$$
$$k_0 r'' = x''$$

$$\therefore\ r^2 \frac{d^2}{dr^2} g(r) + r \frac{d}{dr} g(r) + k_0^2 r^2 g(r)$$

$$\left(\begin{aligned} \frac{d}{dr} g(x/k_0) &= \frac{d g(x/k_0)}{dx} \frac{dx}{dr} = k_0 \frac{dg}{dx}(x/k_0) \\ \frac{d^2}{dr^2} g(x/k_0) &= k_0^2 \frac{d^2 g}{dx^2} \end{aligned} \right)$$

$$\Rightarrow \quad x^2 \frac{d^2 g(x/k_0)}{dx^2} + x \frac{d g(x/k_0)}{dx} + x^2 y = 0 \quad (\alpha = 0)$$

is the homo D.E.

$$\Rightarrow g(x/k_0) = a\, H_0^{(1)}(x) + b\, H_0^{(2)}(x)$$

or $g(r) = a\, H_0^{(1)}(k_0 r) + b\, H_0^{(2)}(k_0 r)$

↳ Great. Now how to go further?

Now we invoke boundary conditions.
The 2D (Sommerfeld) Radiation Boundary condition is:

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial g}{\partial r} + jkg \right) = 0$$

(for $e^{j\omega t}$ time convention).

As $r \to \infty$ $H_0^{(1)}(kr) \to \sqrt{\frac{2}{\pi k r}}\, e^{j(kr - \pi/4)}$

and $H_0^{(2)}(kr) \to \sqrt{\frac{2}{\pi k r}}\, e^{-j(kr - \pi/4)}$

Also $\frac{\partial}{\partial r} H_0^{(1)}(kr) \sim jk\, H_0^{(1)}(kr)$

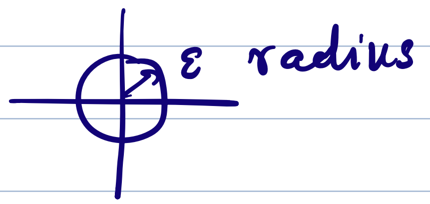
and $\frac{\partial}{\partial r} H_0^{(2)}(kr) \sim -jk\, H_0^{(2)}(kr$

$\Rightarrow$ $H_0^{(2)}(kr)$ satisfies the RBC. $\Rightarrow a = 0$

[Intuitively, we see that $e^{j(\omega t - kr)}$ is the physical soln for a source at origin.] check

$\therefore \quad g(r) = b\, H_0^{(2)}(kr)$ where b is a const to be det.

Now take the full D.E & integrate on a small circle around the origin:



$$\iint_{S_\varepsilon} \left[\nabla^2 g + k_0^2 g = -\delta(r)\right] ds$$

Skipping the detailed steps now, we will get

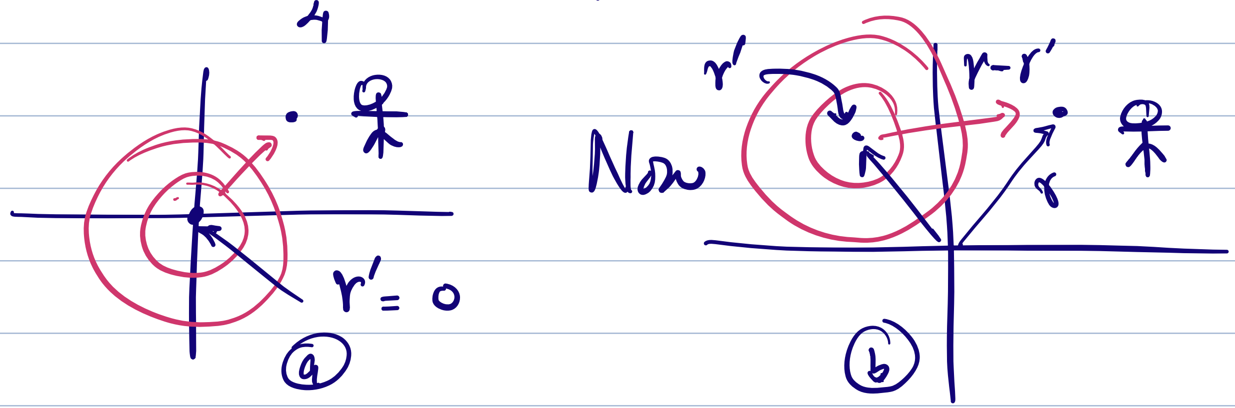
$$\iint_{S_\varepsilon} \nabla^2 g \, ds = -4jb$$

$$\iint_{S_\varepsilon} k_0^2 g \, ds = 0 \quad \& \quad \text{RHS} = -1$$

[see extra pdf for this]

$$\Rightarrow \quad b = -j/4.$$

$\Rightarrow \quad g(r) = -\dfrac{j}{4} H_0^{(2)}(kr)$. Let's get back to (r, r')



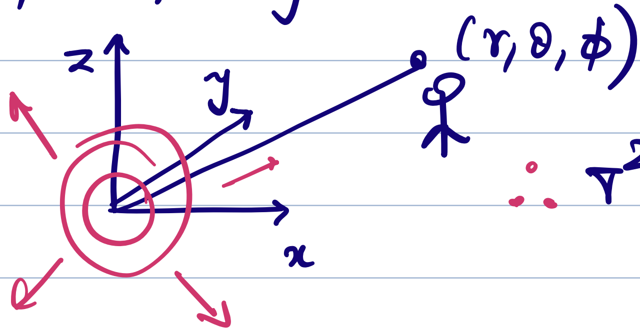
for (b), its as though r' is the origin of the wave.

$$\therefore \quad g(r, r') = -\frac{j}{4} H_0^{(2)}(k\,|r - r'|). \; ✓$$

—x—

↳ Now the 3D wave eqn Green's fn.

Follow a similar strategy of placing r' at 0 to start with.
⇒ No θ, ϕ depn of the soln:



$$\therefore \nabla^2 \rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)$$

∴ Green's fn defn: $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}g(r)\right) + k_0^2 g = -\delta(r)$

Open it $\frac{1}{r^2}\left[r^2 g'' + 2r g'\right] + k_0^2 g = -\delta(r)$

Homogeneous soln: $\frac{1}{r}\left[\underbrace{r g'' + 2g'}\right] + k_0^2 g = 0$

looks like $\frac{d^2}{dr^2}(rg)$

$\therefore \frac{d^2}{dr^2}(rg) + k_0^2 rg = 0$

Looks like the wave eqn.

$\Rightarrow rg(r) = a e^{jkr} + b e^{-jkr}$.

Now the Sommerfeld radiation B.C. states

$$\lim_{r\to\infty} r\left(\frac{\partial g}{\partial r} + jkg\right) = 0$$

This gives $a = 0$, also expected intuitively.

$\Rightarrow g(r) = b\,\dfrac{e^{-jkr}}{r} \rightarrow$ spherical plane wave.

↳ Again, how to determine 'b'? Integrate over a small ball around the origin.

We get $b = 1/4\pi$.

↳ Like before we shift from $r' = 0$ to an arbitrary r'

Finally. $g(r, r') = \dfrac{1}{4\pi}\,\dfrac{e^{-jk|r-r'|}}{|r-r'|}$.

This is the form of Green's fn seen a lot in antenna problems in free space.

Unlike 1D (needed a ∞ sheet of current) or 2D (needed a ∞ line of current) here in 3D we need a tiny point of current → more physically realizable and intuitive.

Summary:

1D	2D	3D
$\dfrac{-j}{2k}\,e^{-jk\|r-r'\|}$	$\dfrac{-j}{4}\,H_0^{(2)}(k\|r-r'\|)$	$\dfrac{1}{4\pi}\,\dfrac{e^{-jk\|r-r'\|}}{\|r-r'\|}$