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Original eqn? $L\phi(r) = f(r)$. Compare with above eqn to get

$$\phi(r) = \int_{\Omega} f(r')\, g(r, r')\, dr'$$

Compare with LTI response $y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$.

This g: Green's fn or impulse response.

— x —

Now we will derive Green's functions for M.F. in 3 cases → 1D, 2D & 3D to cover all possible dimensions. Assumption: free space. A different $\varepsilon(r)$ will give a different fn.

① One dimensional Green's function. —
[Where might we see this? Line currents, Leaky wave antennas, 1D antenna array beamforming, etc.]

Defn of 1D Green's fn: $\dfrac{d^2 g(x,x')}{dx^2} + k_0^2 g = -\delta(x-x')$. ①

We can use FT tricks. Recall:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k) e^{jkx} dk \Rightarrow \frac{d^2 f}{dx^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} -k^2 F(k) e^{jkx} dk$$

$\therefore \quad f(x) \longleftrightarrow F(k) \qquad f'' \longleftrightarrow -k^2 F$

Thus taking FT of ① gives:

$$\left(-k^2+k_0^2\right) G = -e^{-jkx'}$$

$$\Rightarrow g(x,x') = \left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} \frac{e^{jk(x-x')}}{k^2-k_0^2}\,dk \quad -②$$

Note: trouble at $k=\pm k_0$. Called the 'poles'
Can we take it to a closed form expr?

2 routes → usual way contour integration in the complex plane (prereq)
→ Or, via a limiting case of a low loss medium + some F.T. pairs. (no prereq) ✓

ⓐ Key insight: IFT of $\frac{e^{jk(x-x')}}{k^2-k_0^2}$ doesn't exist in standard form, but if we do partial fractions we can do it.

$$\therefore \frac{1}{k^2-k_0^2} = \frac{1}{2\tilde{k}_0}\left[\frac{1}{k-k_0} - \frac{1}{k+k_0}\right]$$

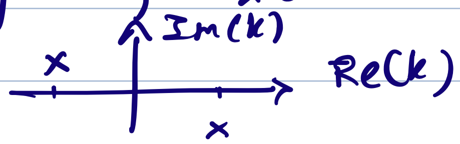
$$\Rightarrow g(x,x') = \frac{1}{4\pi k_0}\int_{-\infty}^{\infty}\frac{e^{jk(x-x')}}{k-k_0}\,dk - \frac{1}{4\pi k_0}\int_{-\infty}^{\infty}\frac{e^{jk(x-x')}}{k+k_0}\,dk$$

ⓑ If the medium has some loss then replace $\tilde{k}_0 = k_0 - j\alpha$, where later we will set $\alpha \to 0$.

So poles are at $\pm (k_0 - j\alpha)$. $\quad \alpha > 0$

[Note: for $e^{-j\omega t}$ we would do $k_0 + j\alpha$]

poles are: $k_0 - j\alpha$, $-k_0 + j\alpha$, $\alpha > 0$



ⓒ Now we use standard FT relations.

$$u(t)e^{-at} \longleftrightarrow 1/a + j\omega \quad \left(\text{when } Re(a) > 0\right)$$

$$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{j\omega t}}{j\omega + a} d\omega = \begin{cases} e^{-at} & t > 0 \\ 0 & t < 0 \end{cases}, \quad Re(a) > 0 \qquad \text{(a)}$$

But this is also correct: OR

$$\begin{cases} 0 & t > 0 \\ -e^{-at} & t < 0 \end{cases}, \quad Re(a) < 0. \qquad \text{(b)}$$

$FT[u(-t)e^{-at}]$ * -1

when $Re(a) < 0$

ⓓ

Look at denominator: $j\omega + a = j(\omega - ja)$

What do we have?

partial fracs $\to$ $k - \tilde{k}_0$ and $k + \tilde{k}_0$

i.e. $ja = \tilde{k}_0$ or $ja = -\tilde{k}_0$

$\Rightarrow a = -j\tilde{k}_0$	$a = j\tilde{k}_0$
$= -j(k_0 - j\alpha)$	$= j(k_0 - j\alpha)$
$= -\alpha - jk_0$	$= \alpha + jk_0$
$Re(a) < 0$	$Re(a) > 0$

Now we know which of eqns (a) & (b) to pick!

$$\therefore \frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{e^{j\omega t}}{\omega-\tilde{k}_0}d\omega = \begin{cases} 0 & t>0 \\ -e^{+j\tilde{k}_0 t} & t<0 \end{cases}$$

$$\frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{e^{j\omega t}}{\omega+\tilde{k}_0}d\omega = \begin{cases} e^{-j\tilde{k}_0 t} & t>0 \\ 0 & t<0 \end{cases}$$

(e) Putting in our language : $\omega \to k$, $x - x' \to t$

$$\frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{e^{jk(x-x')}}{\omega-(k_0-j\alpha)}dk = \begin{cases} 0 & x>x' \\ -\exp(j(k_0-j\alpha)(x-x')) & \text{when } x<x' \end{cases}$$

and

$$\frac{1}{2\pi j}\int_{-\infty}^{\infty}\frac{e^{jk(x-x')}}{\omega+(k_0-j\alpha)}dk = \begin{cases} \exp(-j(k_0-j\alpha)(x-x')), & x>x' \\ 0 & x<x' \end{cases}$$

Back to :

$$g(x,x') = \frac{1}{4\pi k_0}\int_{-\infty}^{\infty}\frac{e^{jk(x-x')}}{k-k_0}dk - \frac{1}{4\pi k_0}\int_{-\infty}^{\infty}\frac{e^{jk(x-x')}}{k+k_0}dk$$

$$\Rightarrow g(x,x') = \begin{cases} \frac{j}{2k_0}\left(-\exp(-jk_0(x-x'))\right) & x>x' \\ \frac{j}{2k_0}\left(-\exp(jk_0(x-x'))\right) & x<x' \end{cases}$$

Can be combined into a single expression:

$$g(x,x') = \frac{-j}{2k_0} \exp(-j k_0 |x-x'|)$$

Makes sense when we toss in $e^{j\omega t}$ time dep.
we get a fwd travelling wave away
from the source at $x = x'$
→ physical interpretation check.

— x —

Note the following properties of Green's fns which hold in general. (Key signatures of g).

1) It satisfies the homogeneous diff eqn.
2) It is symmetric w.r.t x, x'
3) It is continuous at $x = x'$
4) Its derivative is discontinuous at $x = x'$

↓

$$\frac{d^2 g}{dx^2} + k_0^2 g = -\delta(x-x')$$

Integrate over $x = x'_-$ to x'_+

$$\left.\frac{dg}{dx}\right|_{x=x'_-}^{x=x'_+} + k_0^2 \int_{x'_-}^{x'_+} g\, dx = -1$$

i.e. $x = x' - \varepsilon$ to $x' + \varepsilon$. Due to 3,
second term $\to 0$ as $\varepsilon \to 0$

$$\Rightarrow \quad \left.\frac{dg}{dx}\right|_{x'_-}^{x'_+} = -1.$$