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Computing Solutions to Maxwell's eqns

- Given these eqns + Boundary conditions:

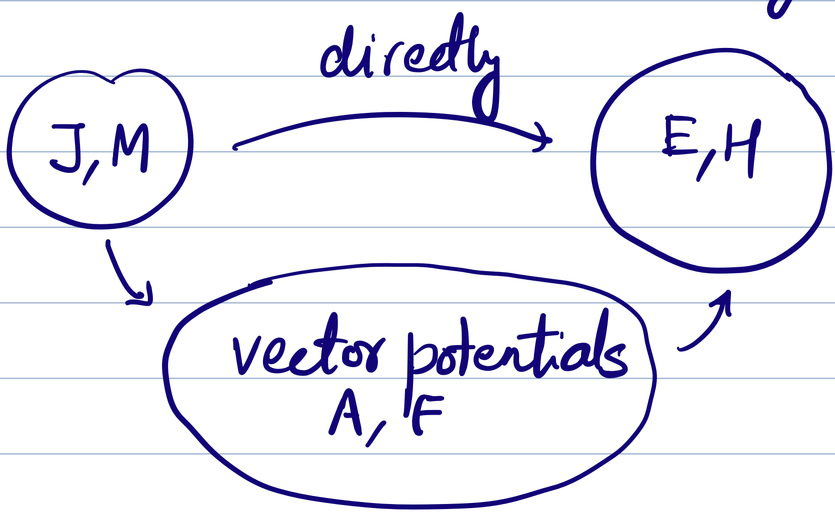
$$\nabla \times \bar{E} = -\bar{M} - j\omega\mu\bar{H}, \quad \nabla \cdot E = \rho_{ev}/\epsilon.$$
$$\nabla \times \bar{H} = \bar{J} + j\omega\epsilon\bar{E}, \quad \nabla \cdot H = \rho_{mv}/\mu.$$

(can be combined into sep eqns for E & H)

① Note: The above is true for homogeneous media, ie. ϵ, μ not fns of $\vec{r}$. Else $\nabla \cdot \vec{D} = \rho_{ev}$.

② Note: $\vec{J}$ can be split as $\vec{J}_{cond} + \vec{J}_{imp}$ where $\vec{J}_{cond}$ is due to the medium and $\vec{J}_{imp}$ is an external / impressed current due to a source.

↳ There are two routes to solving them:



We are familiar with the direct route, so now explore the vector potential route. Assume no charges for now.

① $\underline{\text{First } \vec{A}}$: when $\rho_m = 0$, $\vec{\nabla} \cdot \vec{B} = 0$ (no M)

Vector calculus identity is $\nabla \cdot (\nabla \times \vec{A}) = 0$ for $\underline{\text{any}}$ $\vec{A}$.

Goal? Get an eqn for $\vec{A}$ in terms of $\vec{J}$

Since $\bar{\nabla}\cdot\bar{B}=0$, $\vec{B}$ must be of the form:

$\vec{B}_A = \nabla\times\vec{A} = \mu \bar{H}_A$.

So $\vec{H}_A = \frac{1}{\mu}\nabla\times\vec{A}$ — ①

↳ Then $\nabla\times E = -j\omega\vec{B} = -j\omega\nabla\times\vec{A}$

$\Rightarrow \nabla\times[\vec{E}+j\omega\vec{A}] = 0.$

Another vector calculus identity states

$\nabla\times\nabla\phi = 0$ for any scalar ϕ.

$\Rightarrow$ $\vec{E}+j\omega\vec{A}$ must be of the form $[-\nabla\phi_e]$

Convention — potential — electric

$\therefore \vec{E} = -j\omega\vec{A} - \nabla\phi_e$. — ②

↳ Now let's use the other M.E : (Ampere)

$\nabla\times H_A = j\omega\varepsilon E_A + J = \frac{1}{\mu}\nabla\times(\nabla\times\vec{A})$

$\Rightarrow \nabla(\bar{\nabla}\cdot\bar{A}) - \nabla^2\bar{A} = j\omega\mu\varepsilon\bar{E}_A + \mu\bar{J}$ — ③

Substitute for $\vec{E}_A$ from ② :

$$\nabla(\bar{\nabla}\cdot\bar{A}) - \nabla^2\vec{A} = \omega^2\mu\varepsilon\vec{A} - j\omega\mu\varepsilon\nabla\phi_e + \mu\bar{J}$$

$$\nabla^2\bar{A} + \omega^2\mu\varepsilon\vec{A} = -\mu\bar{J} + \underbrace{j\omega\mu\varepsilon\nabla\phi_e + \nabla(\nabla\cdot A)}_{\nabla(\nabla\cdot A + j\omega\mu\varepsilon\phi_e)}$$

Notice that for $\bar{A}$, so far we have only specified $\nabla\times\vec{A}$ ($= \vec{B}$), but not $\nabla\cdot\bar{A}$.

Based on the Helmholtz decomp thm in vector calculus that holds for fields that decay at ∞, $\vec{A}$ is specified if you specify both its curl & div. $\Rightarrow$ we are free to specify $\nabla\cdot A$. We set $\nabla\cdot A = -j\omega\mu\varepsilon\phi_e$
(Lorentz Gauge condn)

$$\Rightarrow \nabla^2\bar{A} + \omega^2\mu\epsilon\vec{A} = -\mu\vec{J} \quad — (4)$$

(Where have we seen this eqn?)

$$\& \quad \bar{E}_A = -\nabla\phi_e - j\omega A = -j\omega\vec{A} - \frac{j}{\omega\mu\varepsilon}\nabla(\nabla\cdot A). \quad — (5)$$

$$\text{or} \quad \bar{E}_A = \frac{1}{j\omega\mu}\nabla\times\bar{H}_A \quad — (6)$$

↳ So given $\vec{J} \rightarrow$ find $\vec{A} \rightarrow$ find $\vec{H}_A \rightarrow$ find $\vec{E}_A$.

<eqn (4) is the critical step>

(2) The second vector potential, $\vec{F}$. (no J)

When $\rho_e = 0$, $\nabla\cdot D = 0$.

$\Rightarrow$ D must be of curl form.

$$\text{Set} \quad \bar{D}_F = -\nabla\times F \quad — (7)$$

↳ Convention.

$$\Rightarrow \bar{E}_F = -\frac{1}{\varepsilon}\nabla\times\vec{F} \quad — (8)$$

Then $\nabla\times\bar{H}_F = j\omega\varepsilon\bar{E}_F \Rightarrow [H_F + j\omega F] = -\nabla\phi_m$

ϕ_m : scalar magnetic potential.

$$\therefore \quad H_F = -\nabla\phi_m - j\omega\vec{F} \quad — ⑨$$

Going back to ⑧, $\nabla \times E_F = -\frac{1}{\varepsilon}\left(\nabla\times(\nabla\times F)\right)$

$$\Rightarrow -M - j\omega\mu H_F = -\frac{1}{\varepsilon}\left(\nabla(\nabla\cdot F) - \nabla^2 F\right)$$

Like before we have the freedom to choose

$$\nabla\cdot F = -j\omega\mu\varepsilon\phi_m \quad — ⑩$$

Gives $\quad \nabla^2\bar{F} + \omega^2\mu\varepsilon\bar{F} = -\varepsilon\vec{M} \quad — ⑪$

& $\quad H_F = -j\omega F - \frac{j}{\omega\mu\varepsilon}\nabla(\nabla\cdot F) \quad — ⑫$

or $\quad H_F = \frac{-1}{j\omega\mu}\nabla\times E_F \quad — ⑬$

So, given $\vec{M}$ → solve ⑪ to get $\vec{F}$ → find $\vec{E_F}$ from ⑧

↓

find $\vec{H_F}$ from ⑬.

<Solving eqn ⑪ is key>

Note: The critical eqns ④, ⑪ are of the same form, Inhomogeneous wave eqn (RHS ≠ 0).

∴ Net soln if we have both $\vec{J}$ & $\vec{M}$?

$E = E_A + E_F, \quad H = H_A + H_F \quad$ superposition.

<HW: Read solved example 6.1 of AE>

Using this framework all the examples of wave propagation, waveguide modes etc can be expressed in the A, F language.

Note: even if $J = 0$, $M = 0$, we can have both A & F $\neq 0$, in that case:

$$\nabla^2 \vec{A} + \omega^2 \mu \varepsilon \vec{A} = 0 \quad \& \quad \nabla^2 \vec{F} + \omega^2 \mu \varepsilon \vec{F} = 0.$$

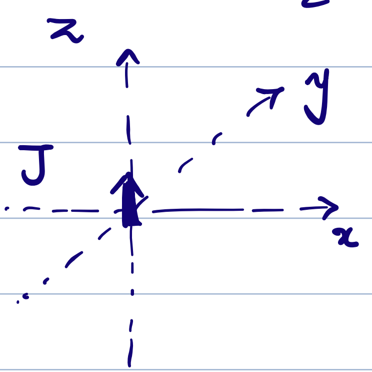
This is the standard wave eqn which we know how to solve!

↳ Inhomogeneous wave eqn, e.g. $\vec{J} \neq 0$.

For simplicity assume $\vec{J} = J_z \hat{z}$, so the wave eqn becomes:

$$\left.\begin{array}{l} \nabla^2 A_x + k^2 A_x = 0 \\ \nabla^2 A_y + k^2 A_y = 0 \end{array}\right\} \text{ easy}$$

$$\left.\nabla^2 A_z + k^2 A_z = -\mu J_z \right\} \text{ not so easy!}$$



This is the standard & most basic antenna problem.

We will come back to it after learning how to solve the z-eqn!

↓

Green's functions / Impulse response
(phy) (EE)

Recall $x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

LTI system.

In time domain, $y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$ convolution.

where $h(t)$ was the impulse response.

How do we get $h(t)$? Recall FT:

$X(\omega) = \int_{-\infty}^{\infty} x(t)\exp(j\omega t)dt$.

Then $Y(\omega) = H(\omega)X(\omega)$.

Set $x(t) = \delta(t) \Rightarrow X(\omega) = 1$

$\Rightarrow Y(\omega) = H(\omega)$.

Only need to do this once. Later any new $x(t) \longrightarrow X(\omega)$ ~~and~~ get $Y = HX$.

Keep in mind H was how the system responded to a delta fn. We borrow this idea next.

— x —

Consider a diff eqn, eg. $\nabla^2\phi = f$ or $\nabla^2\phi + \nabla\phi = f$, etc. Here, f : output and h : input. Write in general as

Operator $\longleftarrow$ $L\phi(r) = f(r) \longrightarrow$ o/p

i/p

↳ What we ideally want is $\phi(r) = L^{-1} f(r)$.
But how?
Define the impulse response, g as follows:

$$L\, g(r, r') = \delta(r, r')$$

only a fn of r.

fn of r & r'

delta input at $r = r'$

How does this help us? Left multiply $f(r')$

$$f(r') L\, g(r, r') = f(r')\, \delta(r, r')$$

can take it inside as L depends only on r.

$$\therefore\ L\, f(r') g(r, r') = f(r')\, \delta(r, r').$$

Now integrate or r' coordinates (both sides) and we must make sure $r' = r$ is in the domain of integration Ω.

$$\int_\Omega L\, f(r')\, g(r, r')\, \underbrace{dr'}_{} = \int_\Omega f(r')\, \delta(r, r')\, dr'$$

can be dx', $dx'dy'$, $dx'dy'dz'$ depending on the problem.

$$\Rightarrow L \int_\Omega f(r') g(r, r')\, dr' = \begin{cases} f(r), & \text{if } r = r' \in \Omega \\ 0 & \text{if } r = r' \notin \Omega. \end{cases}$$

Original eqn? $L\phi(r) = f(r)$. Compare with above eqn to get

$$\phi(r) = \int_{\Omega} f(r')\, g(r, r')\, dr'$$

Compare with LTI response $y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$.

This g: Green's fn or impulse response.