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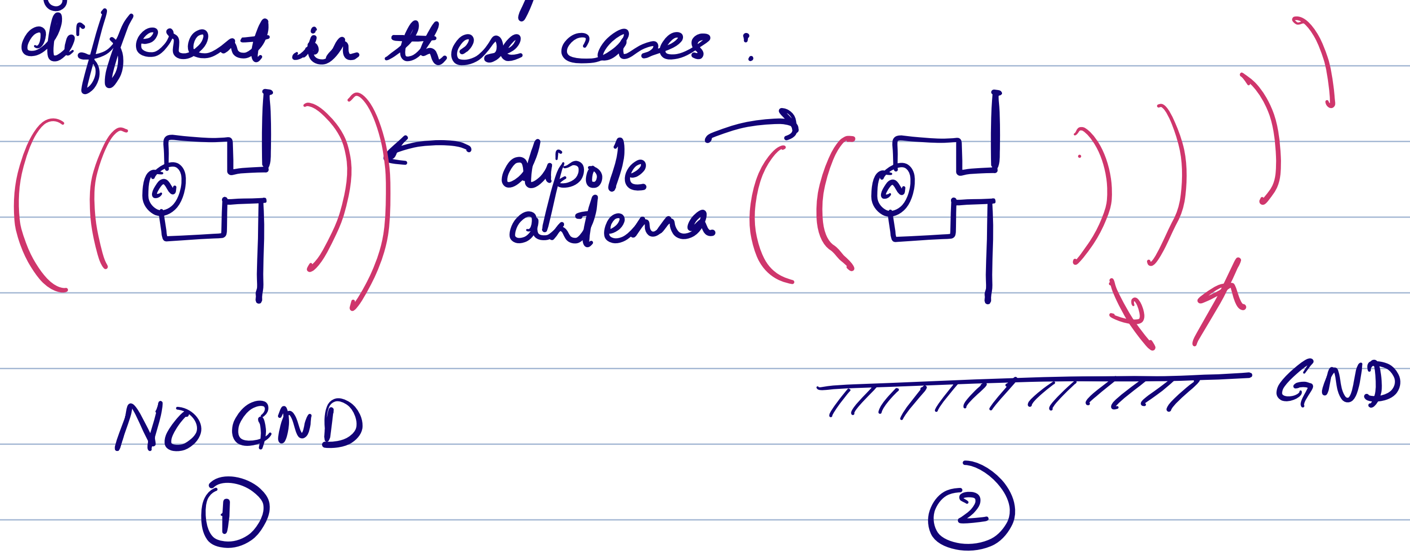
Q: What gives a complete solution to Maxwell's eqns?

① Maxwell's eqns (including sources)
② Boundary conditions

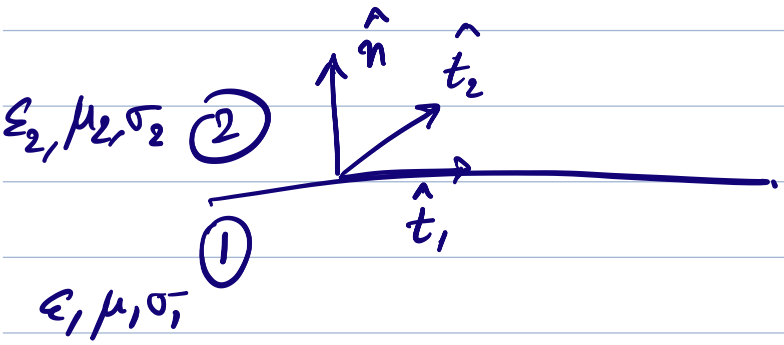
e.g. the radiation patterns, and currents are different in these cases:



↳ We now focus on boundary conditions but from a general ~~starting~~ point to drive home the fundamental concept.

Say the D.E's are: $\nabla \times \vec{P} = \vec{Q} + \vec{R}$, $\nabla \cdot \vec{P} = S$

{where we can interpret $\bar{P}, \bar{Q}$ as fields and $\bar{R}, S$ as an external, or "impressed" quantities (like current, charge)}



$\hat{n}$: normal
$\hat{t}_1, \hat{t}_2$: tangentials.

Recall, how does conductivity feature in ME?

$\nabla \times \bar{H} = j\omega \bar{D} + \bar{J}$. By Ohm's law: $\bar{J} = \underbrace{\sigma \bar{E}}_{\text{in a medium.}}$

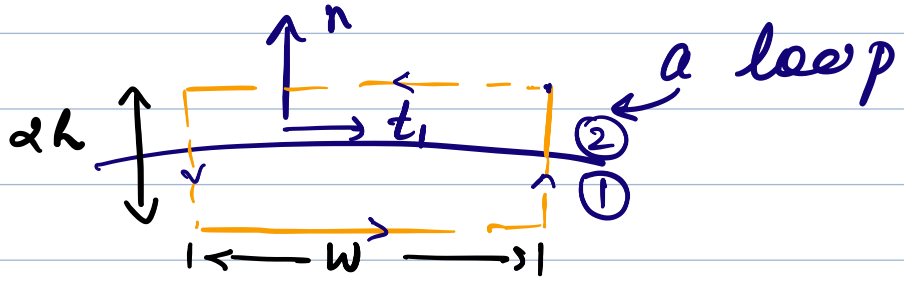
$$\Rightarrow \nabla \times H = j\omega \tilde{\varepsilon}_r \varepsilon_0 \bar{E} + \sigma \bar{E}$$

$$= j\omega\varepsilon_0 \underbrace{\left[\tilde{\varepsilon}_r + \frac{\sigma}{j\omega\varepsilon_0} \right]}\bar{E}$$

$$\varepsilon_c = \varepsilon_r \varepsilon_0 = \varepsilon' - j\varepsilon'' = \varepsilon_0 \tilde{\varepsilon}_r (1 - j\tan\delta)$$

$\hookrightarrow$ complex relative permitivity.

Back to the B.C derivation



Recall divergence thm: $\iint_S (\nabla \times \vec{P}) \cdot d\vec{s} = \oint_\Gamma \bar{P} \cdot d\bar{l}$

$S \hookrightarrow$ open surface

$\Gamma \rightarrow$ its boundary.

LHS:

$$\iint (\nabla \times \bar{P}) \cdot ds \approx -(\bar{Q} + \bar{R}) \cdot \hat{t}_2 (2hw), \text{ if } h, w \text{ small.}$$

RHS: (start from bottom left)

$$P_{1,t_1} w + P_{1,n} h + P_{2,n} h - P_{2,t_1} w - P_{2,n} h - P_{1,n} w$$

strategy: keep w finite small, but $h \to 0$.

↳ <u>case 1 (simple)</u> Q, R are finite qtys, i.e. well behaved physical qtys.

$\Rightarrow$ LHS $\to 0$ as $h \to 0$

$\therefore$ RHS $= 0 \Rightarrow \quad P_{1,t_1} = P_{2,t_1}$

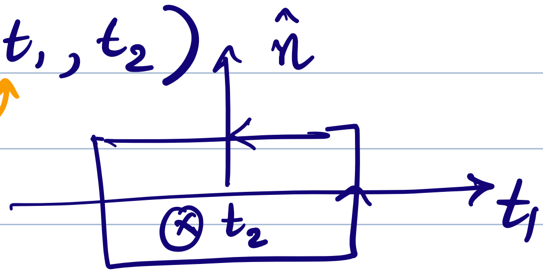
Analogously $\quad P_{1,t_2} = P_{2,t_2}$

Combined as $\quad P_{1,t} = P_{2,t} \Rightarrow n \times (\bar{P}_1 - \bar{P}_2) = 0$

↳ <u>case 2</u>, Q is finite (a field qty) but R is a pure surface current. i.e.

$\vec{R}(n, t_1, t_2) = \delta(n)\, \vec{R}_s(t_1, t_2)$

units?



$\therefore \iint (\bar{Q} + \bar{R}) \cdot d\vec{s}$

↳ along $(-\hat{t}_2)$ if loop is c.clock. (Right hand thumb rule)

$= Q_{t_2}(2hw) +$

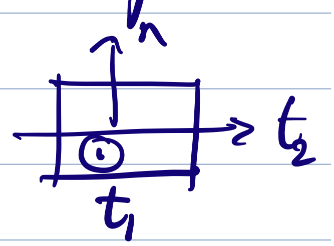
$\iint dn\, dt_1 \left[\vec{R}_s(t_1, t_2) \cdot -\hat{t}_2\right] \delta(n)$.

Use $\int \delta(n)\, dn = 1$ and set $h \to 0$

$\Rightarrow \quad P_{1,t_1} - P_{2t_1} = -\vec{R}_s(t_1, t_2) \cdot \hat{t}_2 \qquad$ (w cancels)

Doing this carefully along the $n - t_2$ loop will give a similar expr:

$P_{1,t_2} - P_{2,t_2} = \vec{R}_s(t_1, t_2) \cdot \hat{t}_1$



What is $\hat{n} \times P_1$?

$$\begin{pmatrix} \hat{t}_1 & \hat{t}_2 & \hat{n} \\ 0 & 0 & 1 \\ P_{t_1} & P_{t_2} & P_{n_2} \end{pmatrix}$$

$= (-P_{t_2}, P_{t_1}, 0)$

$\therefore \hat{n} \times (P_2 - P_1)$

$= ((P_{1,t_2} - P_{2,t_2}), -(P_{1t_1} - P_{2t_1}), 0)$

$= (\bar{R}_s \cdot t_1, \bar{R}_s \cdot t_2, 0)$

Finally, $\hat{n} \times (\bar{P}_2 - \bar{P}_1) = \vec{R}_s$

(R_s has no $\hat{n}$ component, obv)

e.g. $\nabla \times \bar{H} = j\omega D + J$

<u>Case 1</u>: BC is $\hat{n} \times (\bar{H}_2 - \bar{H}_1) = 0$

<u>Case 2</u>: BC is $\hat{n} \times (\bar{H}_2 - \bar{H}_1) = \vec{J}_s$

Surface electrical current.

We will generalize $\nabla \times \vec{E}$ as

$\nabla \times E = -j\omega\mu H - M$ Magnetic current

<u>Case 1</u>: BC is $\hat{n} \times (\bar{E}_2 - \bar{E}_1) = 0$

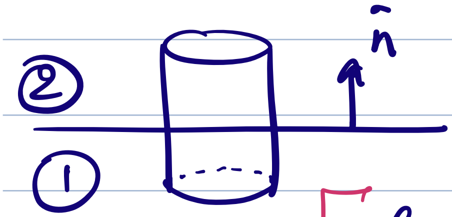
<u>Case 2</u>: BC is $\hat{n} \times (E_2 - E_1) = -\vec{M}_s$

Commonly expressed as:

$$\hat{n} \times (\bar{H}_2 - \bar{H}_1) = \bar{J}_s$$

$$(\bar{E}_2 - \bar{E}_1) \times \hat{n} = \bar{M}_s$$

↳ Using a similar flow and logic as above, we can get BC for normal qtys also., this time using Divergence thm.



$$\oiint \vec{P} \cdot \vec{ds} = \int_V \bar{\nabla} \cdot \bar{P} \, dv$$

Leads to : $\hat{n} \cdot (D_2 - D_1) = \rho_{es}$

and

$\hat{n} \cdot (B_2 - B_1) = \rho_{ms}$

Purely surface qtys { electric charge, magnetic charge

↳ only found in the case of perfect conductors.

↳ Note: All the BCs are not independent of each other. If tangential E, H satisfy the BCs, the normal components also do so.

eg. consider $\hat{n} = \hat{y}$ & $\vec{E}$ in x, z plane. (purely tangential). Then $\nabla \times \bar{E} = -j\omega \bar{B}$

$$\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & 0 & E_z \end{bmatrix} = \left(\frac{\partial}{\partial y} E_z , \frac{\partial}{\partial z} E_x - \frac{\partial}{\partial x} E_z , -\frac{\partial}{\partial y} E_x \right)$$

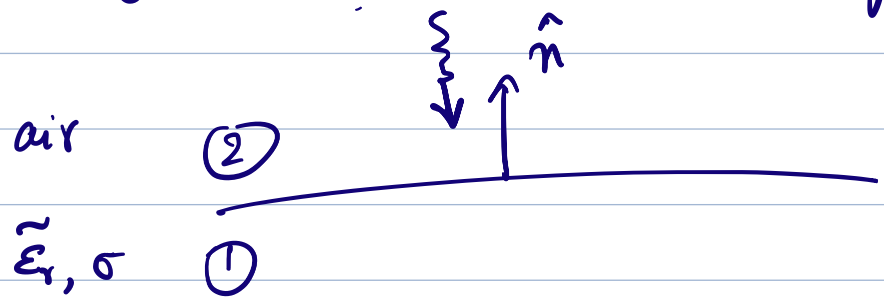
$$= -j\omega (B_x, B_y, B_z)$$

$$\Rightarrow B_y = \frac{j}{\omega} \Big[\underbrace{\frac{\partial}{\partial z} E_x - \frac{\partial}{\partial x} E_z} \Big]$$

These E_{tan} comps are continuous across y

B_y is also conts. $\Leftarrow$

↳ A special BC for metallic structures with high σ, i.e. low skin depths δ.



① Propagation in a medium ($\varepsilon_r \neq 1$) gave a TEM mode (as usual) with
$\left|\frac{E_1}{H_1}\right| = Z_1 = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_r \mu_0}{\varepsilon_r \varepsilon_0}}$. Now,

$$\varepsilon_r = \left(\tilde{\varepsilon}_r + \frac{\sigma}{j\omega\varepsilon_0}\right), \text{ is complex} \Rightarrow Z \text{ complex.}$$

This will be true at any pt in the medium. Similarly in air, for a single travelling wave, $\left|\frac{E_2}{H_2}\right| = Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$.

② ~~At~~ the interface, $(E_1)_{tan} = (E_2)_{tan}$ &

$(H_1)_{tan} = (H_2)_{tan}$.

③ The impedance boundary condn simply states that $\frac{|E_t|}{|H_t|} = Z_1$ at the interface which is not strictly true but found to be a good approx for good conductors.

More explicitly, $\frac{|E_t|}{|H_t|} = z_s \approx \sqrt{\frac{\omega\mu}{2\sigma}}\,(1+j)$

Called a B.C. because tan fields are related.

(4) This is called the impedance boundary condition. Very useful in modelling of metallic structures e.g. antennas, as we dont need to mesh the interior.

$\delta \approx$ μm in case of Cu at GHz

$\lambda \approx$ cm in case of GHz

mesh $\approx \frac{\lambda}{10}$ in numerical tools.

Approximation good under 2 conds:

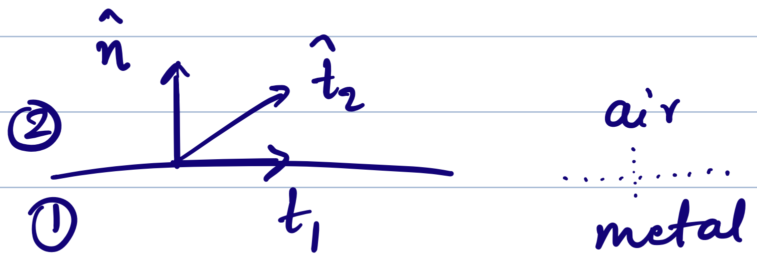
(1) δ very small ~ gets closer to a true sheet current

(2) Normal incidence.

All commerecial tools use this IBC. Higher order versions have been derived.

— x —

ⓐ



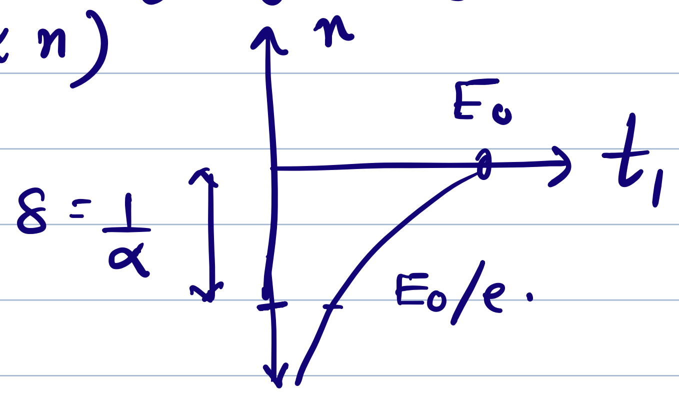
Say $\bar{E}$ along $\hat{t}_2$, $\bar{H}$ along $\hat{t}_1$,
wave going along $-\hat{n}$. ↓

Say at $n=0$, $|\bar{E}| = E_0$.

$\therefore \; E(n,t) = E_0 \exp[(\alpha + j\beta)n] \exp[j\omega t]$

↳ wave going along $-\hat{n}$ dir.

$\Rightarrow |E(n)| = E_0 \exp(\alpha n)$



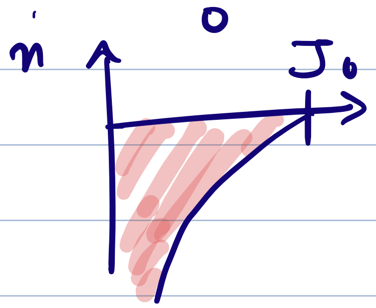
By Ohm's law $J = \sigma E$

& $|J(n)| = \sigma E_0 \exp(\alpha n)$

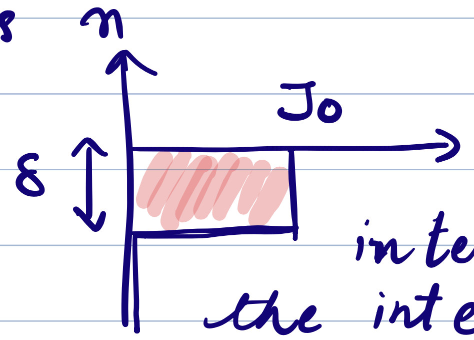
ⓑ Now get an equivalent picture,

$$J_T = \int_0^{-\infty} J(n)(-dn) = \int_0^{\infty} \sigma E_0 e^{-\alpha n} dn = \frac{\sigma E_0}{\alpha}$$

$$= \sigma \delta E_0 = \delta J_0 = J_s$$

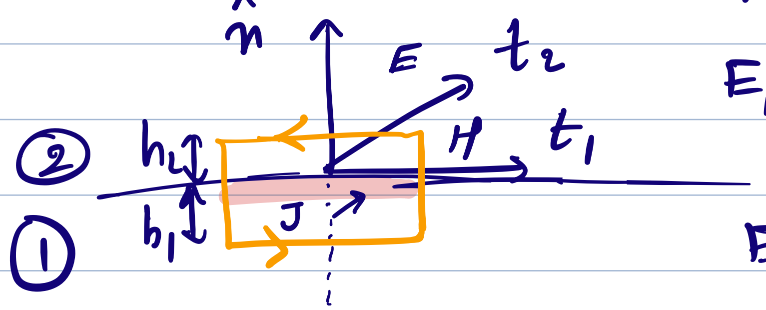


Same as



in terms of the integral eval.

(c) Now imagine a loop in the $n \times t_1$ plane.



Choose $h_1 > \delta$, assume $H_1 \approx 0$ by this value

Then $\hat{n} \times (\bar{H}_2 - \bar{H}_1) = \vec{J}_s \approx \hat{n} \times \bar{H}_2$

↓ correct units!

A/m, $\delta J_0 \rightarrow$ A/m^2

(d) In vector form, $\frac{|E_t|}{|H_t|} = Z_s$

becomes:

$$\bar{E}_t = (\hat{n} \times \bar{H}_t) Z_s$$

, then from (c)

This becomes

$$\vec{E}_t \approx Z_s \vec{J}_s$$

Another form in which IBC is stated.

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