$\phi(x_1, x_2) \to$ fn of 2 variables.

$\|y - Ax\|_2^2$

Searching for minima — for convenience const term = 0

At a critical pt — $\phi(0,0) = 0$. — (1)

$\hookrightarrow \frac{\partial \phi}{\partial x_1} = 0 = \frac{\partial \phi}{\partial x_2}$ — (2)

$$\Rightarrow \phi(x_1, x_2) = \frac{1}{2}\left[\frac{\partial^2 \phi}{\partial x_1^2} x_1^2 + 2 \frac{\partial \phi}{\partial x_1} \frac{\partial \phi}{\partial x_2} x_1 x_2 + \frac{\partial^2 \phi}{\partial x_2^2} x_2^2\right]$$

$$= \frac{1}{2}\Big[\underbrace{a x_1^2 + 2b x_1 x_2 + c x_2^2}\Big]$$

called a quadratic form.

Is $(0,0)$ a minima?

Must always be > 0, i.e $\phi(x_1, x_2) > 0 \ \forall x$

then $\phi(x_1, x_2)$ is called POSITIVE DEFINITE.

$\phi(x_1, x_2) > 0 \longrightarrow$ POS DEF

$\geq 0 \longrightarrow$ POS SEMI DEF

$< 0 \longrightarrow$ NEG DEF

$\leq \longrightarrow$ NEG SEMI DEF

$\cancel{>0}, \cancel{<0} \longrightarrow$ INDEFINITE. (SADDLE).

$$\boxed{a x_1^2 + 2b x_1 x_2 + c x_2^2} = a\left(x_1 + \frac{b}{a} x_2\right)^2 + \left(c - \frac{b^2}{a}\right) x_2^2$$

In order for > 0

$$\Rightarrow \quad a > 0 \;, \quad c > b^2/a \longrightarrow ac > b^2$$

$$\updownarrow$$

$$\left[\frac{\partial^2 \phi}{\partial x_1^2} > 0 \qquad \frac{\partial^2 \phi}{\partial x_2^2} \cdot \frac{\partial^2 \phi}{\partial x_1^2} > \left(\frac{\partial^2 \phi}{\partial x_1 \partial x_2} \right)^2 \right]$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\|y\|^2 = y^T y$$

$$\|y - Ax\|_2^2$$

$$= (y - Ax)^T (y - Ax)$$

$$= y^T y - y^T A x - x^T A^T y + x^T A^T A x$$

A

Always a symmetric matrix.

Defn: A sym matrix A is POS DEF IF $\underbrace{x^T A x}_{\text{A pure quadratic form}} > 0 \quad \forall\, x \neq (0,0)$

Different tests for PD matrices

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \qquad a > 0, \quad ac > b^2$$

1) $|A| = ac - b^2 > 0$

$\lambda_1 \lambda_2 > 0$

$tr(A) = \Sigma \lambda_i$

$a + c = \lambda_1 + \lambda_2 \qquad \because a > 0, \Rightarrow c > 0$

$\therefore \lambda_1, \lambda_2 > 0$

2) $\begin{bmatrix} a & b \\ b & c \end{bmatrix} \longrightarrow \begin{bmatrix} a & b \\ 0 & c - b^2/a \end{bmatrix}$

$R_2 - \frac{b}{a} \times R_1$

Pivots are +ve !

$\rightarrow$ Eigenvalues. $Ax = \lambda x$. x^T L.mult

$$x^T A x = \lambda x^T x = \lambda \|x\|^2$$

- if A is PD $\Rightarrow$ $x^T A x > 0 \Rightarrow \boxed{\lambda > 0}$

If λ's $> 0 \overset{?}{\Longrightarrow}$ A is PD.

Spectral thm $\Rightarrow$ A must have n orthonormal eig vectors.

Take any $x \in \mathbb{R}^n$ $\{x_i\}$

$\Rightarrow \quad x = \sum_{i=1} c_i x_i$ (defn of basis)

A

$$\downarrow$$

$$x^T \swarrow \quad Ax = \sum_{i=1}^{n} c_i \lambda_i x_i \searrow x^T$$

$$\downarrow$$

$$\left[x^T A x = \left(\sum_{i=1}^{n} c_i x_i^T \right) \left(\sum_{j=1}^{n} c_j \lambda_j x_j \right) \right.$$

$$= \sum_{i=1}^{n} c_i^2 \lambda_i \underbrace{\|x_i\|^2}_{=1} > 0 \text{ when } \lambda_i > 0 \Big]$$

$\Rightarrow$ If λ's are +ve, A is PD.

[A real sym matrix A is PD iff eigvals are +ve.]

$\hookrightarrow$ Pivot test. (Recall A is sym).

$A = LU = LDU = LDL^T$

have 1s on diag.

$$x^T A x = \underbrace{x^T L}_{y^T} D \underbrace{L^T x}_{y} = y^T \underset{\substack{\downarrow \\ \text{diag}(D_{ii})}}{D} y$$

$$= [y_1 \cdots y_n]^T \begin{bmatrix} D_{11} & & & 0 \\ & D_{22} & & \\ & & \ddots & \\ 0 & & & D_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

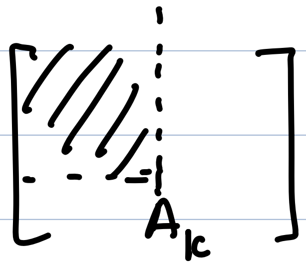
$$x^T A x = \sum y_i^2 D_{ii}$$

If A is PD $\Rightarrow x^T A x > 0 \Rightarrow D_{ii} > 0$

If $D_{ii} > 0 \Rightarrow x^T A x > 0$ $\therefore$ PD

A real sym matrix A is PD iff the pivots are +ve.

↪ Test 3 $\longrightarrow$



All upper left sub matrices have $|A_k| > 0$

Test 4 $\longrightarrow$. A sym matrix A is PD iff there is a matrix R with indepn cols s.t. $A = R^T R$.

1) Assume $A = R^T R$. $\overset{?}{\Longrightarrow}$ A is PD ✓

$$x^T A x = x^T R^T R x = \|Rx\|_2^2 \geq 0$$

But R has indepn cols

$$\therefore \|Rx\|^2 > 0$$

2) Assume A is PD $\overset{?}{\Longrightarrow}$ $A = R^T R$

$$A = L D L^T \quad \text{diag}(D) = D_{ii} > 0 \ (\text{test } 2)$$

$$D = \sqrt{D}\sqrt{D}$$

$$= \underbrace{L\sqrt{D}}_{R^T}\ \underbrace{\sqrt{D}L^T}_{R}$$

$$R = \sqrt{D}\, L = \begin{bmatrix} \sqrt{d_{11}} & & & \\ & \sqrt{d_{22}} & & \\ & & \ddots & \\ & & & \sqrt{d_{nn}} \end{bmatrix} \begin{bmatrix} 1 & & e & \\ 0 & 1 & & \\ 0 & & 1 & \\ & & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{d_{11}} & \sqrt{d_{11}}\, e & \sqrt{d_{11}}\, e \\ 0 & \sqrt{d_{22}}\, e & \sqrt{d_{22}}\, e \\ 0 & 0 & \sqrt{d_{33}} \\ 0 & 0 & 0 \\ & \smile & \end{bmatrix}$$

$\Rightarrow$ R has lin indep cols.

PSD

1) Eigvalues $\lambda_i \geqslant 0$

2) Pivots $D_{ii} \geqslant 0$

3) Submatrices $|A_k| \geqslant 0$

4) A is $R^T R$ with possibly depn cols.

— x —.

Simplest case $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ set $b = 0$

$$x^T A x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$$

$$a x_1^2 + c x_2^2 = 1.$$

$\therefore$ Ellipse in general if A is PD