Proof: $A : n \times n$ $\boxed{A = BRB^H}$

1. $A b_1 = \lambda b_1 \rightarrow$ Atleast 1 eigvector guaranteed!

Say λ_1 has geom. multiplicity $\gamma_A(\lambda_1) = k$.

Take those eigvecs $\rightarrow$ Perform G.S.

$\rightarrow$ Resulting eig vecs : $\{ b_1, b_2, \ldots, b_k \}$

$$A \begin{bmatrix} b_1 & b_2 & \cdots & b_k \\ | & | & & | \end{bmatrix}_{n \times k} = \begin{bmatrix} b_1 & b_2 & \cdots & b_k \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \end{bmatrix}_{k \times k}$$

2. Vectors $\{ b_i \}_{i=1}^{k}$ span $V_b \in \mathbb{C}^n$

Say $\{ c_i \}_1^{n-k}$ spans $V_c \in \mathbb{C}^n \Rightarrow V_b + V_c = \mathbb{C}^n$

Assume c_i's are after G.S.

$$A\begin{bmatrix} b_1 & \cdots & b_k & c_1 & \cdots & c_{n-k}\end{bmatrix} = \begin{bmatrix} b_1 & \cdots & b_k & c_1 & \cdots & c_{n-k}\end{bmatrix}\begin{bmatrix} \begin{matrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_k\end{matrix} & A_r \\ 0 & A_2 \end{bmatrix}$$

$k \times n-k$

$n-k \times k$

$n-k \times n-k$

(3) $$A c_i = \begin{bmatrix} b_1 & \cdots & b_k & \underbrace{c_1 \;\cdots\; c_{n-k}}\end{bmatrix}\begin{bmatrix}(A_r)_i \\ (A_2)_i\end{bmatrix}$$

i^{th} col

Say we change $\{c_i\}$ to $\{d_i\}$ on RHS.

$\Rightarrow$ Only A_2 changes $(n-k \times n-k)$

$T_2 : \{c_i\}_{n-k} \longrightarrow \{d_i\}_{n-k}$.

i.e.

T_2 corresponds to a L.T.

$\Rightarrow$ It must have atleast 1 eigval & 1 eigvec.

Say that the eigvec $\rightarrow d_1$, eigval : λ_2

$$T_2\left(\left[d_1 \ \underbrace{c_1' \ c_2' \dots c_{n-k-1}'}_{\text{After G.S.}}\right]\right) = \left[d_1 \ c_1' \dots c_{n-k-1}'\right] \times \left[\begin{array}{c|c} \lambda_2 & A_2' \\ \hline 0 & \\ 0 & A_2' \\ \vdots & \end{array}\right]$$

Spans same space as $\{c_i\}$

If λ_2 had geom mult $= l$

i.e. l lin. indepn eigvecs with λ_2

$$T_2\left(\begin{bmatrix} d_1 & \cdots & d_\ell & c_1' & \cdots & c_{n-k-\ell}' \end{bmatrix}\right)$$

$$= \begin{bmatrix} d_1 & \cdots & d_\ell & c_1' & \cdots & \underbrace{c_{n-k-\ell}'}_{\substack{\| \\ p}} \end{bmatrix} \left[\begin{array}{c|c} \overset{\ell \times \ell}{\lambda_2 \searrow} & A_{r3} \\ \hline 0 & A_3 \end{array}\right]$$

↪ Full picture:

$$A\underbrace{\begin{bmatrix} b_1 & \cdots & b_k & d_1 & \cdots & d_\ell & c_1' & \cdots & c_p' \end{bmatrix}}_{B'} = B' \begin{array}{c} \\ k \\ \ell \\ p \end{array} \left[\begin{array}{c|c|c} \overset{k \times k}{\lambda_1 \ddots \lambda_1} & A_{r_1} & A_{r_2} \\ \hline 0 & \overset{\ell \times \ell}{\lambda_2 \ddots \lambda_2} & A_{r3} \\ \hline 0 & 0 & \underset{p \times p}{A_3} \end{array}\right]$$

$$\underbrace{\overset{\longleftrightarrow}{k} \quad \overset{\longleftrightarrow}{\ell} \quad \overset{\longleftrightarrow}{p}}_{\text{will end in } R}$$

$\Rightarrow \quad AB = BR$

By construction, B is orthogonal

$\Rightarrow B^{-1} = B^H$

$\Rightarrow \quad A = BRB^H$

Positive Definite Matrices

$$y = \begin{bmatrix} 1 & t_1 & t_1^2 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \longleftrightarrow \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} 1 & t_1 & t_1^2 \\ \vdots & & \\ 1 & t_m & t_m^2 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

$$\Downarrow$$

$$y \overset{?}{=} Ax$$

$e = y - Ax$ error vector.

$$\hat{x} = \boxed{\underset{x}{\operatorname{argmin}} \left(\| Ax - y \|_2^2 \right)} \rightarrow$$ A, b fixed, x tune

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\| Ax - y \|_2^2 = \phi(x_1, x_2) \rightarrow$ quadratic exp. for any no of variables.

Say $\phi(x_1) = \phi(0) + \phi' x_1 + \phi'' \frac{x_1^2}{2} + \cdots$

$\phi' = 0$, $\phi'' > 0 \Rightarrow x_1$ minima

$\phi'' < 0 \Rightarrow x_1$ maxima

$\phi'' = 0 \Rightarrow$ inflection

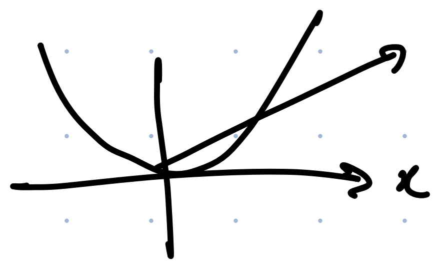
In 2D

$$\phi(x_1, x_2) = \phi(0,0) + \frac{\partial \phi}{\partial x_1} x_1 + \frac{\partial \phi}{\partial x_2} x_2 + \frac{1}{2}\left[\frac{\partial^2 \phi}{\partial x_1^2} x_1^2 + 2 \frac{\partial \phi}{\partial x_1} \frac{\partial \phi}{\partial x_2} x_1 x_2 + \frac{\partial^2 \phi}{\partial x_2} x_2^2 \right]$$

Want to minimize.

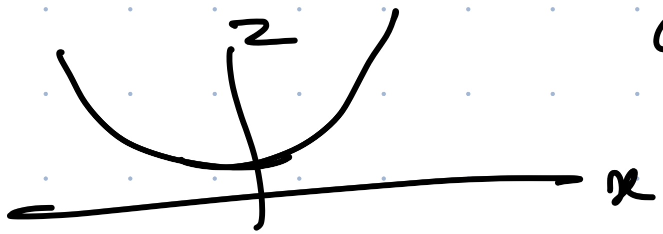
1) Both $\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2} = 0$

eg. $x^2 + y^2$, along x



At $(0,0)$ $\frac{\partial \phi}{\partial x} = 0$, $\frac{\partial \phi}{\partial y} = 0$

if $x^2 - y^2$ along $x \rightarrow$ minima
along $y \rightarrow$ maxima



Saddle point