$|A| = \det(A)$

Review projection matrix.

$v = u + at$

$$\underbrace{\begin{bmatrix}1 & t_1\\ 1 & t_2\\ \vdots & \\ 1 & t_m\end{bmatrix}\begin{bmatrix}u\\ a\end{bmatrix} = \begin{bmatrix}v_1\\ \vdots\\ \vdots\\ v_m\end{bmatrix}}_{Ax = b\ ,\ A \text{ is tall}}$$

$\swarrow$

0 or 1 Soln.

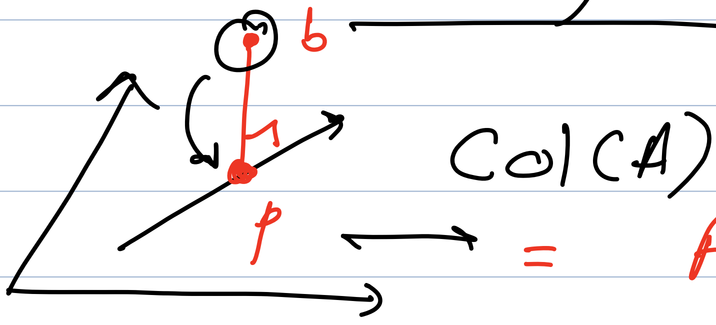
Usually $b \notin Col(A) \Leftrightarrow 0$ Soln.

$\therefore$ Give LSQ soln

$$A^TA\hat{x} = A^Tb$$

Lin Indepn Cols $\Rightarrow$ (A^TA) invertible

$$\underline{\hat{x} = (A^TA)^{-1}A^Tb}$$



$= A\hat{x}$

i/p : b , o/p = p

$$p = \underline{\qquad\quad}\ b$$

$$p = \underbrace{A\left[(A^TA)^{-1}A^T\right]}_{\text{Projection operation.}} b$$

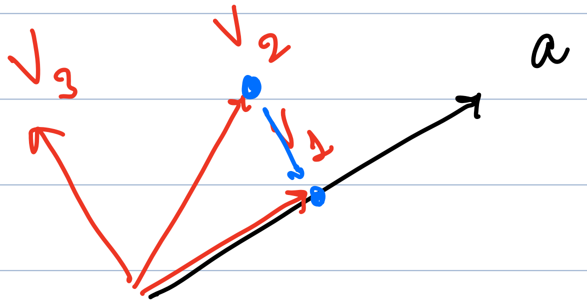
Spl case : A has 1 col.

$A \rightarrow a$

$$\mathbb{P} = a\left((a^Ta)^{-1}a^T\right)$$

$$= \frac{aa^T}{\underbrace{(a^Ta)}_{\text{scalar}}}$$

Geom picture :



$Ax = \lambda x$ → Projection op.

∴ $V_2 \leftarrow$ ✗ eigvec

V_1 · ✓ eigvec $\lambda = 1$

$V_3 \rightarrow$ ✓ eigvec $\lambda = 0$

↳ Eigenspace of A

$$E = \{ x : (A - \lambda I)x = 0 \}$$

↳ Is a vector subspace

∵ null space of a matrix

A subspace of $\mathbb{C}^n$.

→ Alg mult; $\mu_A(\lambda_i) \rightarrow$ How many times λ_i repeats.

Geom mult; $\gamma_A(\lambda_i) \rightarrow$ How many linearly indepn eig-vectors for λ_i

---

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad |A| = \lambda^2 = 0$$

$$\mu_A(0) = 2$$

$$\gamma_A(0) = ?\ 1 \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \overset{\text{same}}{\longleftrightarrow} \begin{pmatrix} j \\ 0 \end{pmatrix}$$

Thm: Given an eigvalue λ_i, we

have:

$$\gamma_A(\lambda_i) \leq \mu_A(\lambda_i)$$

① Let's assume $\gamma_A(\lambda_i) = \gamma$

$\Rightarrow$ $x_1, x_2, \dots, x_\gamma$ ALL have same eigvalue.

② Apply GS to these γ, call them $(x_1, \dots, x_\gamma)$

orthonormal vectros.

③ To make a basis, need $n-\gamma$ more vecs. $\{y_1, y_2, \dots, y_{n-\gamma}\}$

$$S = \begin{bmatrix} x_1 & x_2 & \cdots & x_\gamma & y_1 & & y_{n-r} \\ | & | & & | & | & & | \end{bmatrix}$$

④

$$AS = \left[\begin{array}{cccc|ccc} \multicolumn{4}{c|}{\xleftrightarrow{r}} & \multicolumn{3}{c}{\xleftrightarrow{n-r}} \\ \lambda x_1 & \lambda x_2 & \dots & \lambda x_r & Ay_1 & \dots & Ay_{n-r} \end{array}\right]$$

$n \times n$

$$S^{-1}S = \begin{bmatrix} 1 & 0 & 0 & - & \cdots \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \end{bmatrix}$$

$$\begin{bmatrix} x_1^T & — \\ x_2^T & — \\ \vdots & \\ x_r^T & — \\ \vdots & \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & x_r & y_1 & y_2 & \cdots \\ | & | & & | & & & \end{bmatrix}$$

$$\underline{S^{-1}AS} = \left[\begin{array}{ccc|c} \multicolumn{3}{c}{\leftarrow r \rightarrow} & \\ \lambda & 0 & & \\ & \lambda & & P \\ 0 & & \ddots \; \lambda & \\ \hline R & & & Q \end{array}\right] - cI$$

$$|S^{-1}AS| = |S^{-1}||A||S| = |A|$$

$\Rightarrow$ Char polynomial of A & $S^{-1}AS$

$$p_A(e) = |A - eI| = (e-\lambda_1)(e-\lambda_2)\cdots(e-\lambda_n)$$

$$p_{S^{-1}AS}(e) = |S^{-1}AS - eI| = (e-\lambda)^r q_{n-r}(e)$$

$$= |S^{-1}AS - eS^{-1}S|$$

$$= |S^{-1}(A - eI)S|$$

$$= |A - eI|$$

∴ They have the same eig values.

$$|S^{-1}AS| = \lambda^r q_{n-r}$$

λ is an eigvalue repeated atleast r times.

$$\boxed{\mu_A(\lambda) \geqslant \gamma_A(\lambda).}$$

$$\det(A - eI) = (e-\lambda)^r \underbrace{q_{n-r}(e)}$$

$$= \det(S^{-1}AS - eI)$$

—x—.

Corollaries:

① n×n sys has n eigvalues

$$\sum_{i=1}^{d} \mu_A(\lambda_i) = n$$

②

$$1 \leq \sum_{i=1}^{d} \gamma_A(\lambda_i) \leq n$$

③ Every eigvalue has atleast one eigvector $\Rightarrow \left(\gamma_A(\lambda_i) \geq 1.\right)$

—x—.

A has λ_1, λ_2 and $\lambda_1 \neq \lambda_2$.

$$\begin{tikzcd}
\lambda_1 \arrow[d, leftrightarrow] & \lambda_2 \arrow[d, leftrightarrow] \\
v_1 \arrow[r, leftrightarrow, bend right, "?"'] & v_2
\end{tikzcd}$$

Q:

$$A v_1 = \lambda_1 v_1 \quad ① \qquad A v_2 = \lambda_2 v_2$$

→ $c_1 v_1 + c_2 v_2 = 0$ (only when $c_1 = c_2 = 0$.)

$$A(\quad) = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = 0 \quad ②$$

$\lambda_1 \times ① - ②$, $\lambda_2 ① - ②$

$$c_2 \lambda_1 v_2 - c_2 \lambda_2 v_2 = 0$$

$$c_2 \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \underbrace{v_2}_{\neq 0} = 0 \Rightarrow c_2 = 0$$

$$\Rightarrow c_1 = 0$$

⇒ Distinct eigvalues lead to linearly indepn eig vectors.

— x —