[Q : orthogonal matrix, i.e. cols are orthonormal.]

— . —

Move to the case where $Q \neq$ square.

→ If Q is fat, does it make sense? (the defn)

✗ cols become depn.

→ If Q is tall, does it make sense?

✓ cols stay orthonormal.

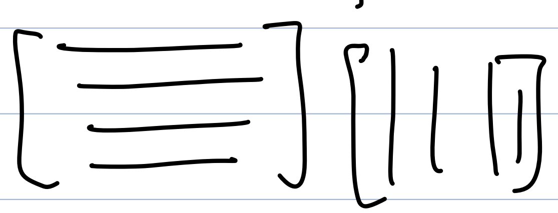
We can have least squared error problems.

Question: are the rows of a tall Q also orthonormal?

No, because rows > no of indepn cols

⇒ some are linearly depn.

$$Q^T Q = I$$



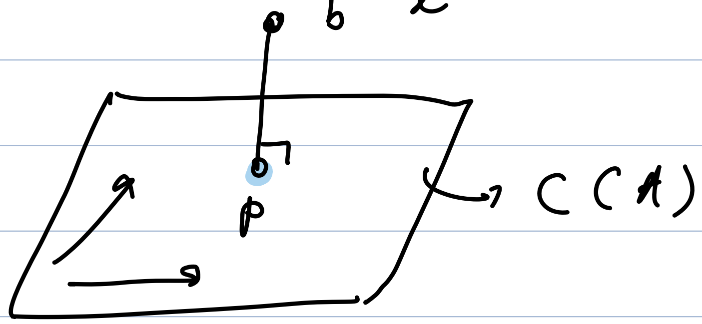
∴ left inverse exist, right doesn't.

Soln to least squared error problem.

$$\hat{x} = Q^T b$$

$$Ax = b$$
$$A^TA\hat{x} = A^Tb$$
$$\boxed{\hat{x} = (A^TA)^{-1}A^Tb.}$$

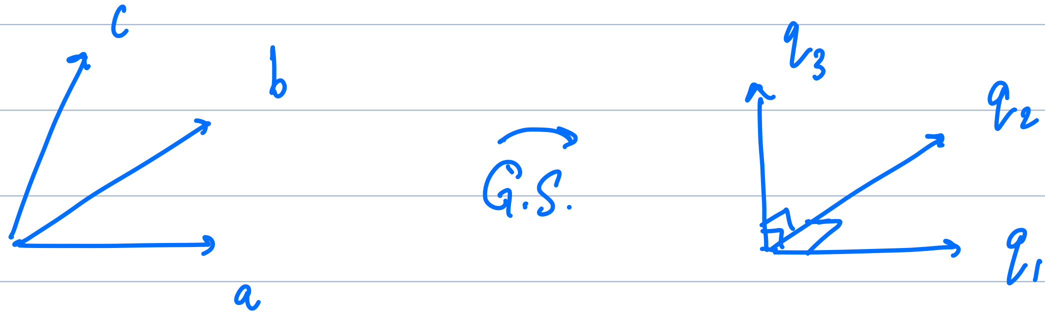
original problem: $\min_x \|Qx - b\|$



$$p = Q\hat{x} = \underbrace{Q\,Q^T}_{P} b$$

P : projection operation.

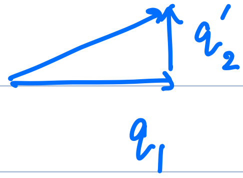
Gram-Schmidt process "constructive"



1) $a \longrightarrow q_1 = a/\|a\|$

2) $b \longrightarrow b' = b - q_1 \underbrace{(q_1^T b)}_{\text{scalar}}$

b



$$q_2 = b' / \|b'\|$$

$q_2 \perp$ to q_1 by construction.

3) $c \rightarrow c' = c - q_1(q_1^T c) - q_2(q_2^T c)$

$$q_3 = c' / \|c'\|$$

$$[a, b, c] \longrightarrow [q_1, q_2, q_3]$$

in general at stage "j" $\rightarrow$

I need to delete $(j-1)$ projections.

$$(a\ \ b\ \ c) = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \rightarrow A$$

1) $q_1' = a = (1\ \ 0\ \ 1)^T$

$\therefore q_1 = \frac{1}{\sqrt{2}}(1\ \ 0\ \ 1)^T$

$$(q_1 q_1^T)c = q_1(q_1^T c)$$

2) $q_2' = b - (q_1^T b)\, q_1$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix} \Rightarrow q_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ \end{pmatrix}$$

3) $$q_3' = c - (\underbrace{q_1^T c}_{\text{scalar}})\underbrace{q_1}_{\text{vector}} - (q_2^T c)\, q_2$$

$$= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \sqrt{2}\,\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \sqrt{2}\,\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}$$

—x—

$\therefore$ order changing $\rightarrow$ different Q.
GS doesn't give a unique set.

#2 Factorization : QR

Q : orthonormal basis vectors.

v

$$v = \sum_{i=1}^{n} c_i q_i$$

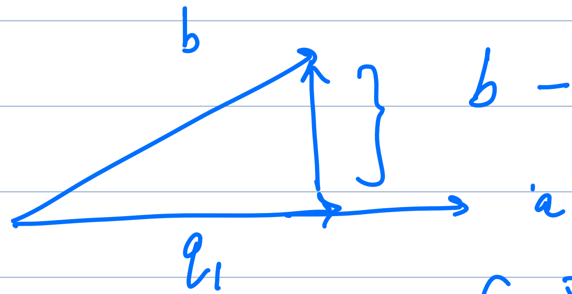
$$\boxed{v = \sum_{i=1}^{n} (q_i^T v)\, q_i}$$

G.S.

→ $a = q_1 \|a\| = q_1 \underbrace{(q_1^T a)}$

$$x^T y = \|x\| \|y\| \cos\theta$$
$$q_1^T a = \|q_1\| \|a\| \cos\theta$$
$$= 1 . \|a\| \, 1 .$$

→ b

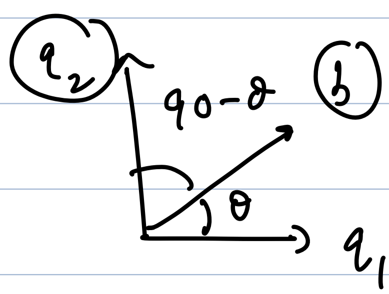


$b - q_1 (q_1^T b)$ scalar vector

$= (q_2^T b) q_2$

$$b = q_1(q_1^T b) + q_2(q_2^T b)$$

$$\therefore \; c = q_1 q_1^T c + q_2 q_2^T c + q_3 q_3^T c.$$



$$b = \underbrace{\boxed{|b| \cos\theta}}_{q_1^T b} \hat{q}_1 + \underbrace{|b| \sin\theta}_{q_2^T b} \hat{q}_2$$

Col pic.

$$\begin{pmatrix} | & | & | \\ a & b & c \\ | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{pmatrix} \boxed{\begin{pmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{pmatrix}}$$

$$\underset{\text{orthogonal.}}{Q} \cdot \underset{\text{Triangular.}}{R}$$

↳ What happens when A is tall

$A_{m\times n}$ $\quad m > n.$

$$\underset{m\times n}{A} = \underset{m\times n}{Q} \; \underset{n\times n}{R}$$

Can always do this as long as:
cols are indepn.
+ R is invertible.

—x—

[What is the advantage of QR in solving least sq. error problems?]

$$Q\underbrace{Rx}_{y} = b$$

$$Qy = b$$

$$\Rightarrow y = Q^T b$$

$$Rx = y \longrightarrow \text{Can be solved by back subst.}$$

 $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \vdots \\ y_n \end{bmatrix}$

No free lunch $\rightarrow$ GS is expensive!

Same complex LU