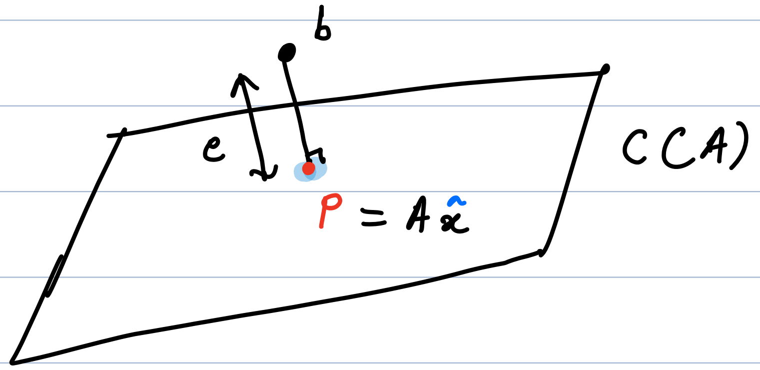
$v = u + at$ $\qquad$ $A_{m\times n}$

① $m > n \Rightarrow$ Either 0 or 1 soln. $b \in C(A)$

② $e = b - Ax \rightarrow$ error vector.

③



p: closest point in C(A) to point b.

: also the projection of b onto C(A).

④ How to derive this point 'p'?

a) $e \perp C(A) \Rightarrow e \in N(A^T)$ (LNS)

Defn of $N(A^T)$? $A^T y = 0$

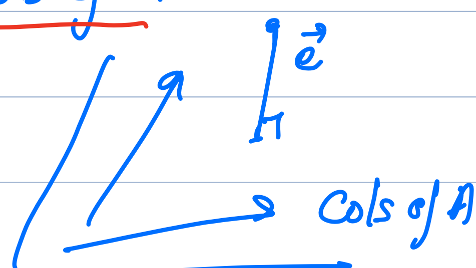
$\Rightarrow A^T(e) = 0 = A^T(A\hat{x} - b)$

$$\Rightarrow \boxed{A^T A \hat{x} = A^T b}$$

b) C(A) spanned by cols of A.

→ e must be ⊥r to all cols of A ←

e^T [——] A $\begin{bmatrix} | & | & | & | \end{bmatrix}$ $\qquad e \begin{bmatrix} | \end{bmatrix}$



$$e^T \begin{bmatrix} \text{———} \end{bmatrix} A \begin{bmatrix} | & | & | & | \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

A_i

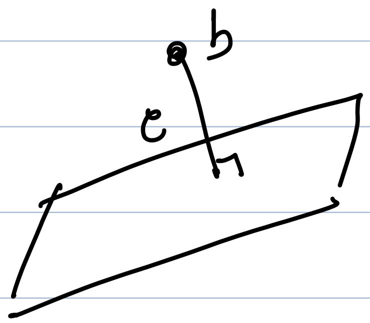
$$\left. e^T A_i = 0 \right\} \Rightarrow e^T A = 0$$

$$A^T e = 0$$

Once again: $A^T(A\hat{x} - b) = 0$

$$\boxed{A^T A \hat{x} = A^T b}$$

(c)



$C(A)$

find $\hat{x}$ s.t.
$\|e\| = \|A\hat{x} - b\|$ is minimized

$$\|e\|_2^2 = \sum_j |e_j|^2 \qquad e = Ax - b$$

$$= \sum_j \Big[\sum_k A_{jk} \hat{x}_k - b_j \Big]^2$$

$$\therefore \frac{\partial \|e\|_2^2}{\partial \hat{x}_i} = 0 \Rightarrow \sum_j 2 \Big[\sum_k A_{jk} \hat{x}_k - b_j \Big] A_{ji} = 0$$

$$\frac{d}{dx} f(x)^2 = 2 f(x) f'(x)$$

$$\nabla \|e\|^2 \qquad \sum_j \underbrace{A_{ji}}\left[\underbrace{\sum_k A_{jk} \hat{x}_k - b_j} \right] = 0$$

$$= \sum_j (A^T)_{ij} \times (A\hat{x} - b)_j = 0$$

row

col.

j^{th} row

This must be true for all i

i^{th} row of $\quad A^T (A\hat{x} - b) = 0$

$$\Rightarrow A^T (A\hat{x} - b) = 0$$

$$\Rightarrow A^T A \hat{x} = A^T b$$

The matrix $A^T A$ appears everywhere!

↪ Square matrix : $n \times n$

↪ Symmetric $\quad (A^T A)^T = A^T (A^T)^T = A^T A$

↪ [If A has independent cols, then $A^T A$ is invertible.

will prove.

In $A \rightarrow$ cols are indepn.

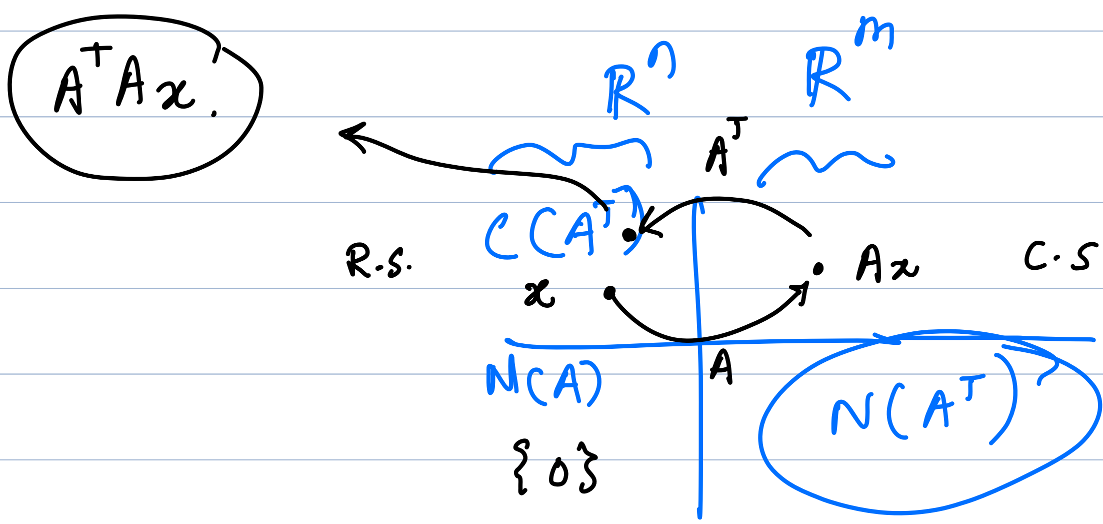
RREF $\rightarrow \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ \hline & & 0 & \end{bmatrix} \; \updownarrow n \quad \updownarrow m$

$\Rightarrow$ all vars are pivot vars.

$\Rightarrow$ No free vars

$\Rightarrow \quad N(A) = \{\vec{0}\}$

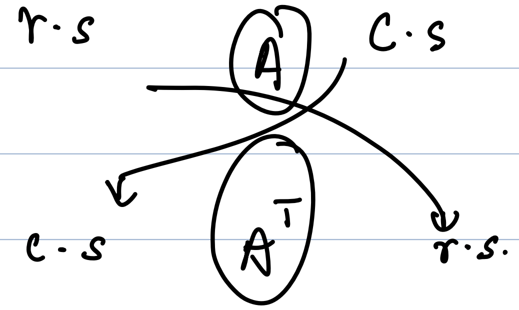
$\Rightarrow \quad C(A^T)$ is the entire $\mathbb{R}^n$.



Intuitively : A^TA is invertible

1) Any x in $\mathbb{R}^n$ goes to c.s. by A

2) Every y in c.s. goes to the r.s by A^T



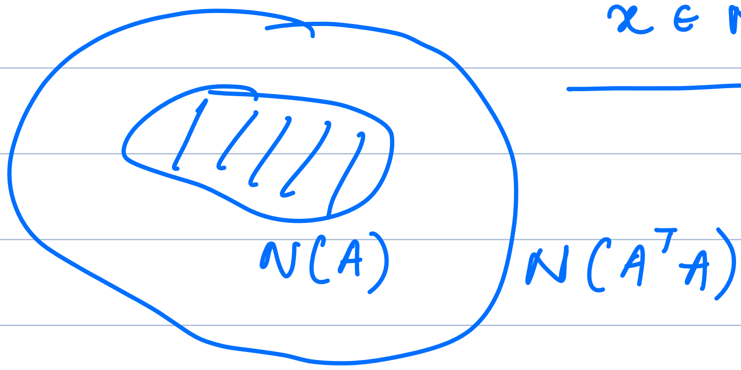
$$\begin{tikzcd} x \arrow[r, "A"] & Ax \arrow[r, "A^T"] & A^TAx \arrow[ll, bend left, "?", "(A^TA)^{-1} \text{ if it exists.}"'] \end{tikzcd}$$

Formal:

① If $x \in N(A)$, then $x \in N(A^TA)$?

why? $Ax = 0 \rightarrow \underbrace{A^T A x = 0}$

$x \in N(A^TA)$.



$$\Rightarrow N(A) \subset N(A^TA).$$

② If $x \in N(A^TA)$, then $x \in N(A)$?

$A^TAx = 0$

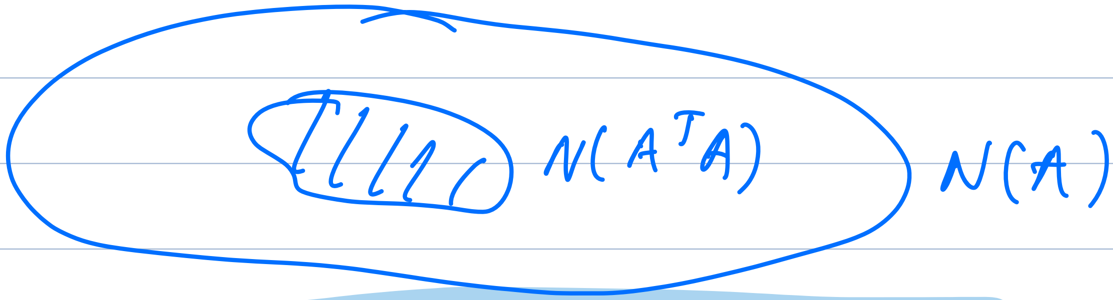
left mult by x^T

$$x^T A^T A x = 0$$

$$\|Ax\|^2 = 0$$

$$\Rightarrow Ax = 0$$

$$\Rightarrow x \in N(A).$$



③ ① & ② $\Rightarrow$ $\boxed{N(A) = N(A^TA)}$

④ A has $m > n$ & independent cols.

$\Rightarrow$ A has col rank $= n$

$\Rightarrow$ A has row rank $= n$.

By rank nulity thm

$$\Rightarrow \dim(r.s.) + \dim(n.s) = n$$

$$\Rightarrow \dim(n.s) = 0$$

$$\Rightarrow N(A) = \{0\}$$

$$\Rightarrow N(A^TA) = \{0\}$$

⑤ Apply rank nulity thm to (A^TA)

$n \times n$

row rank + nullity = n

dim (n.s) = 0

$\Rightarrow$ row rank $(A^T A) = n$

$\Rightarrow$ full rank ($\Leftrightarrow$ row rank = col rank)

$\Rightarrow$ Matrix is invertible

$$\hat{x} = (A^T A)^{-1} A^T b$$

as long as cols of A are lin indepn.

least squared error soln.