2) <u>Orthogonality of Subspaces</u>.

- Every vector in one subspace is orthogonal to every vector in the other subspace

- $\{\vec{0}\}$ and every other VS ✓
- a line & another line ✓
- a line & an orthogonal plane ✓
- Two planes ✗

We have already seen $A: m \times n$

$C(A^T)$, $d = A^T w$, dim $= r$, $\mathbb{R}^n$	$C(A)$, $c = Ax$, dim $= r$, $\mathbb{R}^m$
$N(A)$, $Ax = 0$, dim $n-r$, $\mathbb{R}^n$	$N(A^T)$, $A^T y = 0$, dim $= m-r$, $\mathbb{R}^m$
$d^T x = w^T \underbrace{Ax} = 0$	$c^T y = x^T \underbrace{A^T y} = 0$

Every vector is covered.

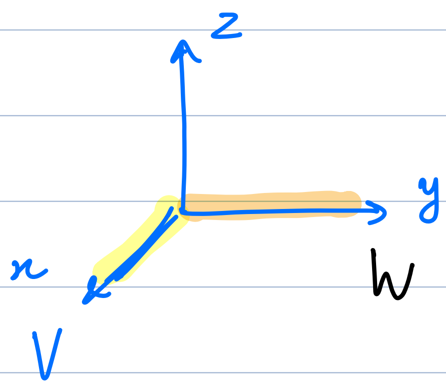
— orthogonal subspaces.

Formally: Given a subspace V, the space of all vectors orthogonal to V is called the orthogonal compliment of V: $V^{\perp}$ ("v perp")

$$C(A) = N(A^T)^{\perp}$$
$$C(A^T) = N(A)^{\perp}$$

Q: Can 2 v.s. be orthogonal without being compliments? ✓

e.g.



V & W are orthogonal.

e.g z axis

$\therefore$ these are not compliments.

- If V is a v.s. of $\mathbb{R}^m$ with dim r, then the compliment must be of dim $m-r$

↳ prove? Basis for V: $\{v_1, \dots, v_r\}$

Put in a matrix

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_r \\ \uparrow & | & & \uparrow \end{bmatrix}$$

$$[\ \ \ ]_{m\times r}$$

Colspace = left N.S$^{\perp}$

$\downarrow$ dim r $\quad\quad$ $\downarrow$ $m-r$ $\quad$ ✓

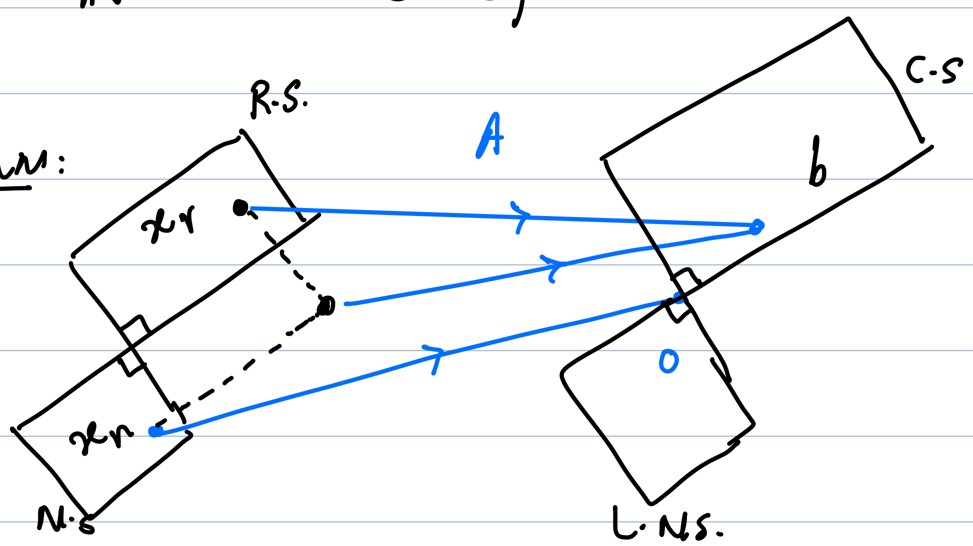
(proof by construction).

- **Bottom line:** Any space $\mathbb{R}^n$ can be broken up into a subspace V & its orthogonal compliment.

$\rightarrow$ $\mathbb{R}^n$ = Row space + Null space.

$\mathbb{R}^m$ = Col space + L. Null space.

Thm:



$$x = x_r + x_{n.s}$$

From row sp to col space, A is invertible.

$\Rightarrow$ for every b in $C(A)$, there is exactly one x_r in $C(A^T)$.

Proof: (by contradiction)

Say $Ax_r = b$, & another x_r'

$Ax_r' = b$, then

$$A(x_r - x_r') = b - b = 0$$

$$\Rightarrow \quad x_r - x_r' \in N(A)$$

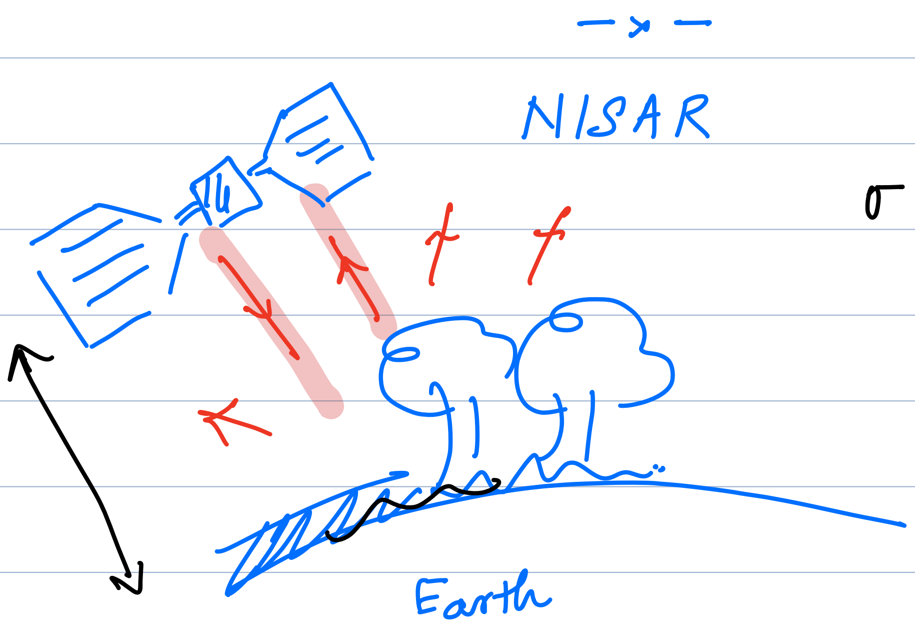
This is a contradiction!

$$Ax_r = b$$

If there exists a pseudo inverse $A^+(Ax_r) = x_r$

$$A^+(Ax_n) = 0$$

— x —



$\sigma(x_1, x_2 \dots, x_n)$

$\downarrow$

soil moisture
veg content
tree cover.

To do: Build a linear model for σ in terms of $\{x_1, .., x_n\}$.

What you have: Radar meas &
access to ground data.

① The model.

$$\sigma = A x + a_0$$

A: $1 \times n$, $x \in \mathbb{R}^n$, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$\sigma(x) = \beta_0 + \sum_{i=1}^{n} \beta_i x_i$$

unknown? β's.

② Determine coefs of model.

Ground data $\rightarrow$ $\{x_1^{(1)}, \ldots, x_n^{(1)}\}$ set of values.

meas $\rightarrow$ σ_1.

$$\underbrace{\begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_{n+1} \end{pmatrix}}_{\sigma} = \underbrace{\begin{pmatrix} 1 & x_1^1 & x_2^1 & x_3^1 & \cdots & x_n^1 \\ \vdots & & & & & \\ 1 & x_1^{n+1} & & \cdots & & x_n^{n+1} \end{pmatrix}}_{A \;\; n+1 \times n+1} \underbrace{\begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}}_{\beta}$$

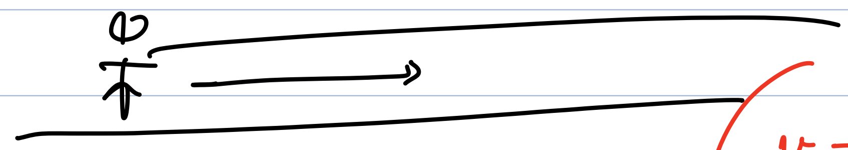
③ Solve: $\beta = A^{-1}\sigma$. if A^{-1} exists.

if doesn't, then a problem.

—x—

Least Squares Problems

Const acc $\quad v = u + at$



$v = u + at + \text{noise}$

$v = (u + at) \times \text{noise}$

At various t, measure v.

Objective: estimate acc., u

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \\ 1 & t_n \end{bmatrix} \begin{bmatrix} u \\ a \end{bmatrix} \Rightarrow Ax = b$$

v_i: can have noise,

t_i: can also have noise.

Assume cols of A are lin indepn.

$m > n$.

more meas than variables.

1) $\underline{m > n}$ $\Rightarrow$ Either one or zero solns.

$b \in C(A)$.

due to errors in meas, most likely

b

$C(A)$

2) Say that $b \notin C(A)$, what is the best that we can do?

error vector : $e = b - Ax$.

We want ? x s.t. $\|e\|$ is minimized.

$$\min_{x} \left[\| b - Ax \| \right]$$

variable of min