- Next tut (2) → 17/2
- tut(1) quiz → in class 14 or 15th Feb.

Linear Transformations.

$\mathbb{R}^n$	$\mathbb{R}^m$
Row sp	Col sp

$$A x = b$$

x in $\mathbb{R}^n$, b → $\mathbb{R}^m$

The matrix A is "moving" x from $\mathbb{R}^n$ to $\mathbb{R}^m$

Rand vect in $\mathbb{R}^n$ → Row sp, → Null sp $\Big\}$ $\mathbb{R}^n$

$x \neq 0$, $x \in$ Row sp., due to A, → goes to Col sp.

Row sp. + Nullsp $\xrightarrow{\text{"}A\text{"}}$ Col sp.

— x —

4 examples.

$$A_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

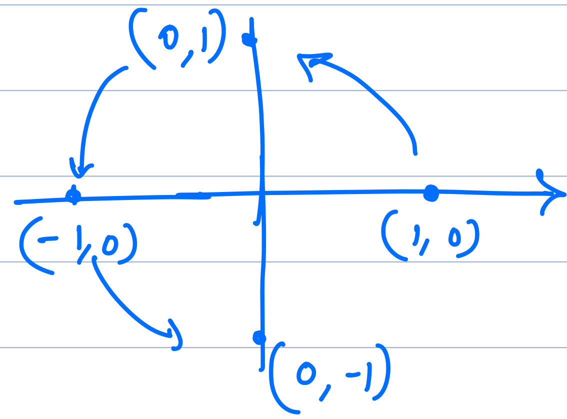
$$A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

input $\begin{pmatrix} x \\ y \end{pmatrix}$

1) scaling, stretching.

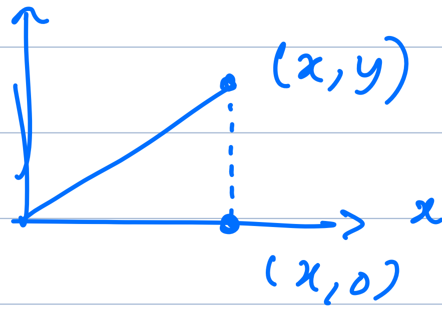
2) $A_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$

Rotation



3) Swap/mirror/reflect.

4) Projection on x axis.



What is common?

↳ working on linear comb of cordinates.

↳ None of them move the origin!

↳ $u \xrightarrow{A} v$

$p \xrightarrow{A} q$

$\alpha u + \beta p \xrightarrow{A} \alpha v + \beta q.$

⇒ Applies to any transf that comes from a matrix.

→ What is a linear transf. (also linear map).

A mapping T between 2 vector spaces V and W that preserves the operations of addition & scalar multiplication.

$\therefore$ a matrix is a linear transformation.

↳ e.g. the space of polynomials, $p(t)$, degree n.

e.g. $p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$

$n+1$ coefs are reqd. $\quad P_n$

Q1: operation of $\frac{d}{dt}$ is linear?

$$\frac{d}{dt}(a_0 + \cdots + a_n t^n) = a_1 + 2a_2 t + \cdots + n a_n t^{n-1}$$

P_n

$1+n$

$p(t) \rightarrow \frac{d}{dt} \rightarrow p'(t)$

$q(t) \rightarrow \frac{d}{dt} \rightarrow q'(t)$

P_{n-1}

$\therefore$ This is a linear transf.

Does this transf have a null space

Yes! any $p(t) = a_0$

$$\left.\begin{aligned} \text{Nulity} &= 1 \\ \text{rank} &= n \end{aligned}\right\}$$

Sum of rank, nulity = dim of orig space P_n

e.g. 2. Integration: $\int_0^t$

$$\int_0^{t'} = \int_0^t p(t)\,dt = \underbrace{a_0 t + \frac{a_1 t^2}{2} + \cdots + \frac{a_n}{n+1} t^{n+1}}$$

Is it a linear transf ✓

No nontrivial null space

↳ Transformations as Matrices.

space of P_n (polynomials).

$$p(t) = a_0 + a_1 t + \cdots + a_n t^n.$$

Coefs: $\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} \hookrightarrow n+1$

Is there a basis to express such a vector?

Yes, canonical basis

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$p_1 \quad p_2 \qquad p_{n+1}$

$$p = \sum_{i=1}^{n+1} a_i p_i \qquad p_1 \to \text{const term.}$$

e.g. of differentiation.

$$\frac{dp(t)}{dt} = a_0 \times 0 + a_1 \times 1 + a_2 2t \cdots + a_n n t^{n-1}$$

$\boxed{\text{basis of } P_3}$ $\quad [p_1 = 1, \; p_2 = t, \; p_3 = t^2, \; p_4 = t^3]$

due to d/dt $\quad \boxed{Ap_1 = 0}$, $Ap_2 = p_1$, $Ap_3 = 2p_2$, $Ap_4 = 3p_3$

$$A[p_1 \; p_2 \; p_3 \; p_4] = A_{diff} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

input space basis vectors

$p = 2 + t - t^2 - t^3$. find $\frac{dp}{dt}$.

$$p = \begin{pmatrix} 2 \\ 1 \\ -1 \\ -1 \end{pmatrix} \qquad ? \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -3 \\ 0 \end{pmatrix}$$

$1 - 2t - 3t^2$ ✓

Recipe.

$$V \xrightarrow{T} W$$

1) Need a basis for the input space $V: \{x_1, \dots, x_n\}$
2) Need a basis for the o/p space $W: \{y_1, \dots, y_m\}$

Col j of A $\quad T(x_j) = (\alpha_1 y_1 + \cdots + \alpha_m y_m)$

e.g. 2 integration $\quad V = P_3, \quad W = P_4$

$[x_1 = 1, x_2 = t, x_3 = t^2, x_4 = t^3]$

$[y_1 = 1, y_2 = t, y_3 = t^2, y_4 = t^3, y_5 = t^4]$

$$A \longrightarrow \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix}}_{A_{integ.}}$$

$1^{st} \to x_1$

$T(x_1) \to t$, in 'y' basis

$T(x_2) \to \frac{t^2}{2}$

$T(x_3) \to t^3/3$

$T(x_4) \to t^4/4$

(we didn't need y_1)

$p(t) = t + \frac{t^2}{5}$ $\quad$ i/p.

$+$

$$\int_0 p(t)\,dt = \frac{t^2}{2} + \frac{t^3}{15} \quad o/p.$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1/5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 1/15 \\ 0 \end{pmatrix}$$

✓

— x —