(Row) Echelon form

1. All 0 rows at the bottom
2. Pivot of any row is strictly to the right of the pivot of the row above it.

$$\overbrace{\begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{pmatrix}}^{\boxed{A}} x = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{\boxed{U}} x = \begin{pmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{pmatrix}$$

$C(A)$: 2 are suff

$v_2 = 3v_1$

$v_4 = -v_1 + v_3$

$$b = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}, \quad b' = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

$Ux = b' \Rightarrow 3x_3 + 3x_4 = 3$

$x_3 = 1 - x_4$

$x_1 + 3x_2 + 3x_3 + 2x_4 = 1$

$x_1 = -2 - 3x_2 + x_4$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \underbrace{\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{particular soln}} + \underbrace{\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \underset{p}{x_2} + \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \underset{q}{x_4}}_{\text{null space part.}}$$

Need a b, i.e RHS vector.

Q: Is this ( ) a vector space?

A: No, because it lacks $\vec{0}$. It is an affine space.

Summarize: A: $m \times n$ matrix

1. If there are r pivots $\Rightarrow$ r pivot variables.
   $\Rightarrow$ free variables $= n - r$
   r : rank of the matrix.

2. $Ax = b \rightarrow Ux = b' \rightarrow Rx = b''$
   Rank of A, U, R have same rank r
   $\Rightarrow$ Rows of U, R that are $0 = m - r$

$\therefore$ For there to be a soln to $Ax = b$
bottom $m - r$ rows of b' and $b'' = 0$

called the Solvability condn

③ To get a particular soln, set all free variables = 0

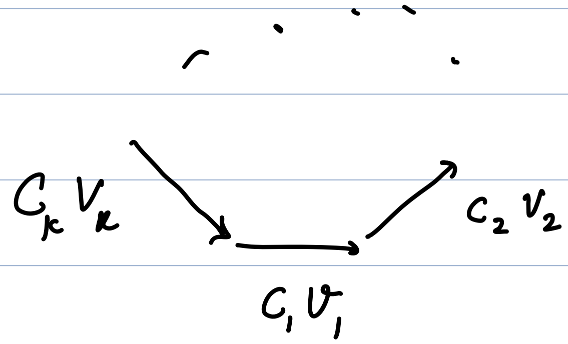
④ Null space is the L.C. of $\underline{n-r}$ vectors

ω Linear independence

k vectors: $\{v_1, \ldots, v_k\}$ are linearly indepn iff a linear combination, i.e.

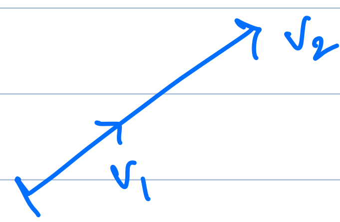
$\boxed{\sum c_i v_i = 0}$ only when $c_1 = c_2 \cdots = c_k = 0$

ALL of them are 0.



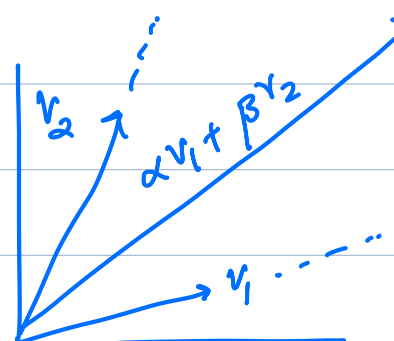
· eg. linearly depn

1) If $\vec{v}_1 \parallel \vec{v}_2$, then $\bar{v}_1 = \alpha \bar{v}_2$



$1 \times v_1 + (-\alpha) v_2 = 0$ ∴ Not indepn.

2) If $\bar{v}_1 \cancel{\parallel v_2}$



$\alpha v_1 + \beta v_2 = 0$

must happen for $\alpha = \beta = 0$

3) v_1, v_2, v_3 ~~is~~ are in the same plane.
say $v_3 = \alpha v_1 + \beta v_2$.

$$\alpha v_1 + \beta v_2 + (-1) v_3 = 0$$

$\therefore$ there are non-zero coffs s.t. $\sum c_i v_i = 0$
$\Rightarrow$ they are dependent.

Say $A \hookrightarrow \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \sum \vec{v}_i x_i$

Defn of linear indepn $\Rightarrow$ $Ax = 0$ iff $x = 0$

$$\Rightarrow N(A) = \{\vec{0}\}$$

Conversly if $N(A) \neq \{\vec{0}\}$,
$\therefore$ there is some x^n s.t $Ax^n = 0$
$x^n \neq \vec{0}$

Cols of A are indepn when $N(A) = \{\vec{0}\}$

$\hookrightarrow$ When we saw $A: m \times n$ $\quad m < n$.

$\begin{bmatrix} & & \end{bmatrix}$ Max pivots $= m$

min free vars $= n - m > 0$

$\Rightarrow N(A) \neq \{\vec{0}\}$

$\Rightarrow$ Cols of A are linearly depn

$\Rightarrow$ [A set of n vectors in $\mathbb{R}^m$ must be linearly dependent if $m < n$.]

↳ Consider the Echelon form (U or R)

3 pivots var.

1 free var

$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \\ \end{array}$$
$$\begin{bmatrix} 1 & x & x & x \\ 0 & 1 & x & x \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1) Rows that are linearly indepn $= 3$

2) Cols corresponding to the pivots are linearly indepn

→ How many in no? 3

→ The r non-zero rows of an Echelon matrix are linearly indepn and so are r pivot cols.

↳ Spanning a space.

If a vector space $\underline{V}$ consists of a L.C. of $\{v_1, \ldots, v_k\}$, then we say that these vectors span the space V.

i.e. $\forall\ v \in V$, we have $v = \sum_{i=1}^{k} c_i \vec{v}_i$

Note: these c_i's are not unique.

In $\mathbb{R}^3$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow$ canonical basis

$\vec{v} = x\hat{x} + y\hat{y} + z\hat{z}$.

Not unique $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$

$v = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$

To make matters more economical, we define a basis, 2 properties

1. The basis vectors should be linearly indep^{n}
2. They should span V.

↳ Uniqueness: Given a set of basis vectors $\{v_1, \ldots, v_k\}$ there is only one way of expressing v in terms of them.

Proof: ? $\vec{v} = \sum a_i \vec{v_i} = \sum b_i \vec{v_i}$

$$\Rightarrow \boxed{0 = \sum (a_i - b_i)\vec{v_i} = 0}$$

But this is a contradiction because $\vec{v_i}$s are independent $\Rightarrow$ all $(a_i - b_i) = 0$

$\Rightarrow a_i = b_i$.

Basis $\longrightarrow$ Not unique

Given basis $\rightarrow$ L.C. is unique.

e.g. $A \rightarrow U = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

1) The cols of U span its col space ✓

2) The cols of U form a basis for col space ✗
(∵ they are linearly depn)

3) $C(A) = C(U)$ ✗
(shown earlier)

↪ Dimension of a vector space.

= . The no of basis vectors.