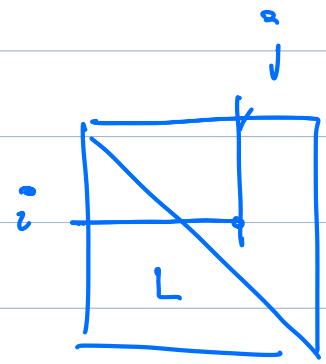
$$\underbrace{E^{-1}F^{-1}G^{-1}}_{?\ L}\ \overbrace{\underbrace{G\,F\,E}_{\text{Row operations}}\ A}^{\text{upper tri}} = A$$

$\Rightarrow$

How to prove: $P \times Q$: P, Q are lower tri

P.T $M = PQ$ is also lower tri.

$$\underset{\text{row}\ \ \text{col}}{M_{ij}} = \sum_{k=1}^{n} P_{ik}\, Q_{kj}$$



$P_{ik} = 0$ if $i < k$

$Q_{kj} = 0$ if $k < j$

look at $\underline{i < j}$

$$= \underbrace{\sum_{k=1}^{j-1} P_{ik}\, Q_{kj}}_{k<j} + \underbrace{\sum_{k=j}^{n} P_{ik}\, Q_{kj}}_{k \geq j > i}$$

$k < j$

$\Rightarrow Q_{kj} = 0$

$k \geq j > i$

$\Rightarrow k > i$

$\Rightarrow P_{ik} = 0$

$= 0$

— x —

$\Rightarrow$ Net: $\quad L \cdot U = A$

called the LU decomposition of A

$$\boxed{\text{When can there be trouble?}}$$

↳ $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$ $\quad$ <u>soln:</u> swap rows

pivoting

↳ $\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ d & e & f \end{pmatrix}$

1) $d = 0 \Rightarrow$ No first pivot
2) $d \neq 0,\ a = 0$
$\Rightarrow$ No second pivot
3) $d \neq 0,\ a \neq 0$
$c = 0 \Rightarrow$ No third pivot

$\Rightarrow$ If any pivot $= 0 \Rightarrow$ No unique soln to $Ax = b$.

singular.

triangular with 1 on diags.

$\rightarrow \quad PA = LU = LDU$

$$U = \begin{pmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} 1 & d/a & e/a \\ 0 & 1 & f/b \\ 0 & 0 & 1 \end{pmatrix}$$

If A is non singular, then $A = LDU$

is unique.

↳ How to use?

$$\left.\begin{aligned} Ax &= b \\ L\underbrace{Ux}_{y} &= b \end{aligned}\right)$$

(1) $Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 & \cdot \\ * & 1 & 0 & 0 \\ * & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \Rightarrow \begin{aligned} y_1 &= b_1 \\ &\vdots \\ y_n &= \text{---} . \end{aligned}$$

FWD subst.

(2) $Ux = y$

$$\begin{bmatrix} \ddots & * \\ 0 & \ddots \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \leftarrow \text{known}$$

$$\Rightarrow \begin{aligned} x_n &= \checkmark \\ &\vdots \\ x_1 &= \checkmark \end{aligned}$$

back subst.

Many engg appl: same system matrix A, several RHS vectors b.

LU : $O(n^3)$.

$$Ax = b$$

1) $GFE = C$

2) $CAx = cb$

$\Rightarrow \quad Ux = \underset{\sim}{Cb}$

Solve.

Effects of Rounding $\leftrightarrow$ Pivoting?

$$0.123400 + 0.005567$$

$= 0.128967$ (to us)

$\to 0.129$ (to comp)

$$A = \begin{pmatrix} 0.0001 & 1 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad x = \begin{pmatrix} u \\ v \end{pmatrix}$$

Q: What is the ans $\nearrow$ to us, $\searrow$ to comp

us

$$\to \begin{pmatrix} 1.0001 \\ 0.9999 \end{pmatrix}$$

comp

$$\begin{pmatrix} 1.00010001 \\ 0.9998\ldots \end{pmatrix}$$

$$\to \begin{pmatrix} \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Step 1: $R_2 \to R_2 - 10000R_1$

$$\begin{pmatrix} 0.0001 & 1 & 1 \\ 0 & -9999 & -9998 \end{pmatrix}$$

$$v = \frac{-9998}{-9999} = 0.9998999 \text{ (to us)}$$

$$= 1 \text{ (to comp)}$$

Back subst

$$u \times 0.0001 + v = 1$$

to the comp $\Rightarrow u = 0$

Comp: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Correct ans? $u \times 0.0001 + 0.999899 = 1$

$$\Rightarrow u = 1$$

us: $\begin{pmatrix} 1 \\ 0.99989 \end{pmatrix}$

To do: Partial pivoting.

Matrix Inverse

Existence → If all pivots are non zero

If any are zero
No existence

Uniqueness : If it exists it is unique.

$Ax = 0$ for $x \neq 0$. does A^{-1} exist?

If it exists $A^{-1}Ax = A^{-1}0$

$\Rightarrow x = 0$

Contradicts

$(AB)^{-1} = B^{-1}A^{-1}$

— x —