$$\begin{array}{c}\begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix} \\ v_1 \quad\;\; v_2 \end{array} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{array}{c}\begin{pmatrix} 11 \\ 14 \\ 17 \\ 20 \end{pmatrix} \\ v_0 \end{array}$$

Row picture
→ 4 lines.

Column picture

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix} \quad \text{— (1)}$$

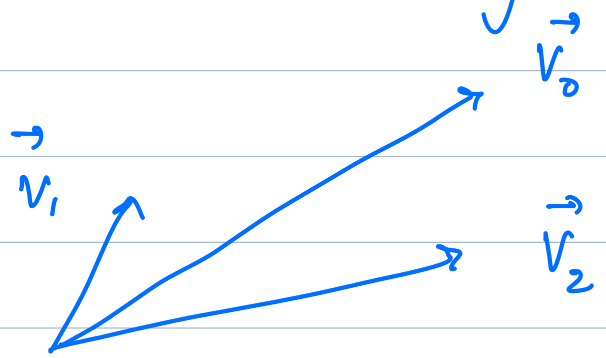
$$\vec{v}_1 \quad \begin{pmatrix} a \\ c \end{pmatrix} x + \begin{pmatrix} b \\ d \end{pmatrix} y = \begin{pmatrix} e \\ f \end{pmatrix} \quad \text{— (2)}$$

$\vec{v}_2$ $\vec{v}_0$

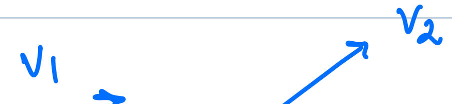
split into cols.

look at a col as a vector.

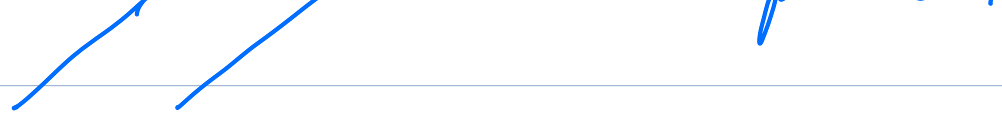
Linear combination of 2 vectors.



"Col picture" of a matrix.



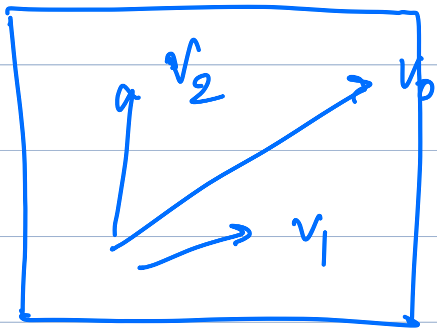
both parallel.



→ No solns

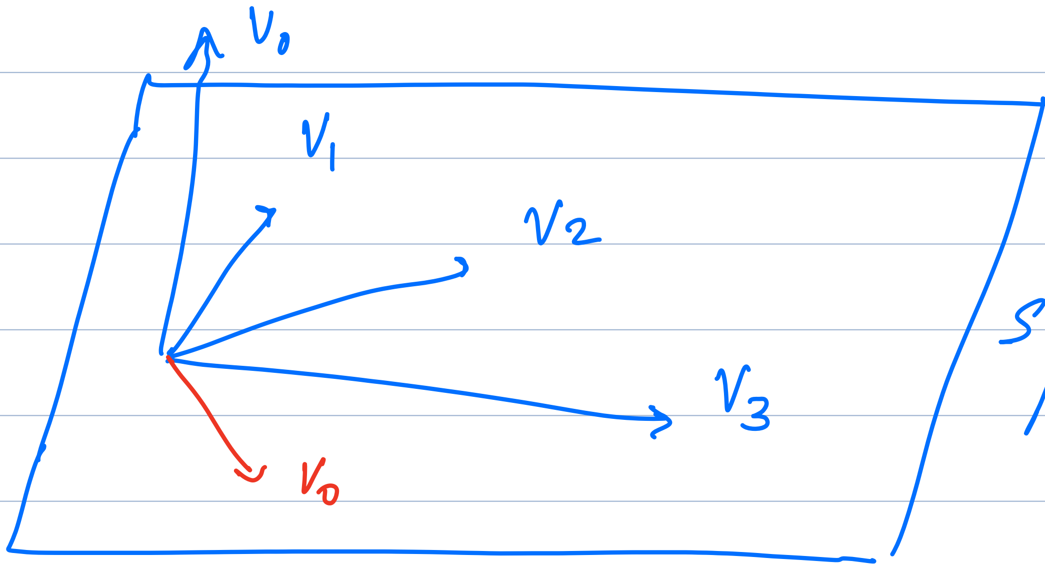
→ ∞ solutions

2D



as long as $v_1 \not\parallel v_2$
then unique soln

3D



same plane

Solns? 0 solns — v_0 out of plane

v_0 in plane → ∞ solns.

vector spaces

—x—

Gaussian Elimination.

$$R_1 \longrightarrow \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix}$$

$R_3 \to R_3 - 7R_1$ Row eliminations

$$\begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d \\ e \\ f \end{pmatrix} \quad \text{back subst.}$$

$$cz = f \implies z = f/c$$

$$bx + * z = e \implies x =$$

Estimate the computational cost:

Step 1: Getting 0s below 1^{st} entry of col 1

$$= n(n-1) = n^2 - n$$

Next $(n-1)(n-2)$

Total: $\displaystyle\sum_{k=1}^{n} k(k-1) = \sum_{k=1}^{n} k^2 - k$

LHS

$$= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$

$$= \frac{n^3 - n}{3} \to O(n^3) \text{ operations}$$

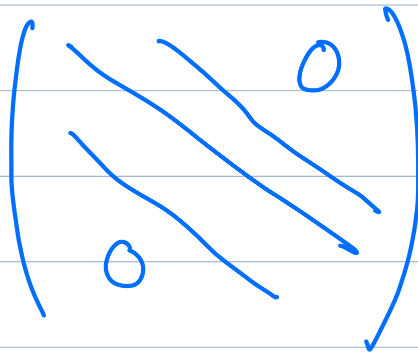
$n = 10^6$

RHS $\to$ $1 + 2 + \cdots + n \sim O(n^2)$

Overall $O(n^3) \to$ Reduced A to a upper triangular matrix.

IBM $\to$ $O(n^{2.376})$

e.g. sparse matrix ⟵ opp ⟶ dense.

$A =$ 

sparse $n_z << N$

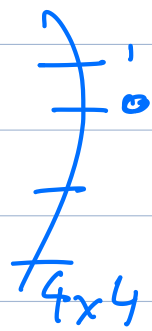
⟶ —

Permutations

$$\underset{P}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

Row swap.

$$\underset{P_{12}}{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \Big(\quad \Big)$$



Rows 1 & 2 get swapped.

13 23

14 24 34 6 ✓

↳ Different matrix

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l & 0 & 1 \end{pmatrix}}_{E} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

We want $R_3 - l R_1 = R_3'$

$$= \begin{pmatrix} a & b & c \\ d & e & f \\ g - la & h - lb & i - lc \end{pmatrix}$$

E_{31} → col no, ↓ Row no

for $R_3 = R_3 - l R_2$, E_{32} needs mod.

E: elementary matrix

1) $R_3 \to R_3 - l R_1$ → $E_{31}^{(1)} = -l$

2) $R_2 \to R_2 - m R_1$ → $E_{21}^{(2)} = -m$

$E^{(2)} E^{(1)} A$

$E \downarrow$

$$\begin{pmatrix} 1 & 0 & 0 \\ -m & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -m & 1 & 0 \\ -l & 0 & 1 \end{pmatrix}}_{E}$$

worked on col 1

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -p & 1 \end{pmatrix}$$

work on col 2.

$$\cdots \underbrace{\overset{G}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -p & 1 \end{pmatrix}} \overset{E}{\begin{pmatrix} 1 & 0 & 0 \\ -m & 1 & 0 \\ -l & 0 & 1 \end{pmatrix}}} A = U$$

F

$$(GFE)A = U \quad — ①$$

$$(E^{-1}F^{-1}G^{-1})(GFE)A = A \quad — ②$$

lower triangular

The (smart) way to get E^{-1} is to undo the row operation!

$E \rightarrow (R_2 - l R_1) \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ -l & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$E^{-1}? \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ +l & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \leftarrow$

$E^{-1}, F^{-1}, G^{-1} \leftarrow$

To do: calc : $E^{-1} \times F^{-1} \times G^{-1} \rightarrow$ what type of matrix

—x—