Constraint Qualification:

Conditions under which linearized feasible set $\mathcal{F}(x)$ is similar to the tangent cone $T_\Omega(x)$.

Recall: CONE a set $\mathcal{F}$ s.t. $\forall\, x \in \mathcal{F}$, $\Rightarrow \alpha x \in \mathcal{F} \quad \forall\, \alpha > 0$

convex 1) e.g. $\{[x_1, x_2]^T \mid x_1 > 0,\ x_2 > 0\}$ is a cone

n. convex 2) e.g. $\{[x_1, x_2]^T \mid x_1 \geq 0, \text{ OR } x_2 \geq 0\}$ is a cone

3) $x = \sum_{i=1}^{m} \alpha_i x_i$ for $\alpha_i \geq 0$ generates a convex cone given $\{x_1, \ldots, x_m\}$

Specific example: common eg. LICQ:
linear independence constraint qualification:
We say LICQ holds if the set of active constraint gradients is linearly independent at some pt x^0 $\{\nabla c_i(x^0),\ i \in \mathcal{A}(x)\}$.

First Order Necessary Condns (FONC)

$\rightarrow$ Karush Kuhn Tucker Condns (KKT).

(As before define $\mathcal{L}(x, \lambda) = f(x) - \sum \lambda_i c_i(x)$)

$i \in \mathcal{E} \cup \mathcal{I}$

is the Lagrangian

If:

1) If x^* is a local soln of $\min_{x \in \Omega} f(x)$,
2) f and c_i's are continuously differentiable
3) LICQ holds at x^*

Then: There is a set of Lagrange multipliers λ^* s.t. the foll. are satisfied at (x^*, λ^*):

a) $\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$

b) $c_i(x^*) = 0, \quad \forall \; i \in \mathcal{E}$

c) $c_i(x) \geq 0, \quad \forall \; i \in \mathcal{I}$

d) $\lambda_i \geq 0, \quad \forall \; i \in \mathcal{I}$

e) $\lambda_i c_i(x) = 0 \quad \forall \; i \in \mathcal{E} \cup \mathcal{I}$.

Complementary condn

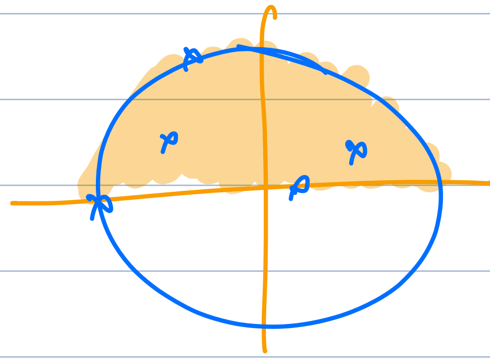
Finally: There can be many pts where a-e holds but with LICQ the optimal λ^* is unique.

Aside: Why no strict inequalities:

e.g. $\min_{x > 0} x$, $\max_{x > 0} 1/x$

eg. $\min_{0 \leq x < 5} x$

Proof sketch.



$$\min f(x) = x_1 + x_2$$

s.t. $2 - x_1^2 - x_2^2 \geq 0$

$x_2 \geq 0$

FONC for x^* to be a local min:

(i) $c_1(x^*) \geq 0, \quad c_2(x^*) \geq 0$

(ii) a) If $c_1(x^*) = 0, \quad c_2(x^*) = 0$, then

$\Rightarrow \nabla f(x^*) - \lambda_1^* \nabla c_1(x^*) - \lambda_2^* \nabla c_2(x^*) = 0$

for $\lambda_1^*, \lambda_2^* \geq 0$

b) If $c_1(x^*) > 0$ and $c_2(x^*) = 0$, then

$\Rightarrow \nabla f(x^*) - \lambda_2^* \nabla c_2(x^*) = 0 \qquad \lambda_2 \geq 0$

c) If $c_1(x^*) = 0$ and $c_2(x^*) > 0$, then

$\Rightarrow \nabla f(x^*) - \lambda_1^* \nabla c_1(x^*) = 0, \qquad \lambda_1 \geq 0$

d) If $c_1(x^*) > 0, \quad c_2(x^*) > 0$, then

$\Rightarrow \nabla f(x^*) = 0$

proof: $\rightarrow$ (i) is trivial because x^* is feasible

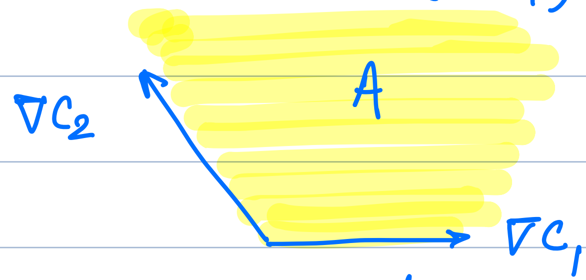
$\rightarrow$ For (ii) A, we need to show

$$\nabla f = \lambda_1^* \nabla c_1(x^*) + \lambda_2^* \nabla c_2(x^*)$$

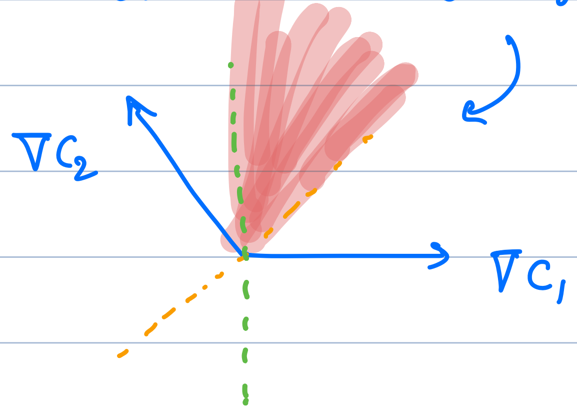
for $\lambda_1^*, \lambda_2^* \geq 0$

$$A : \{ u \in \mathbb{R}^2 : u = \lambda_1 \nabla c_1 + \lambda_2 \nabla c_2 \text{ for } \lambda_1, \lambda_2 > 0 \}$$

$$= \text{cone}(\nabla c_1, \nabla c_2)$$



feasible directions?

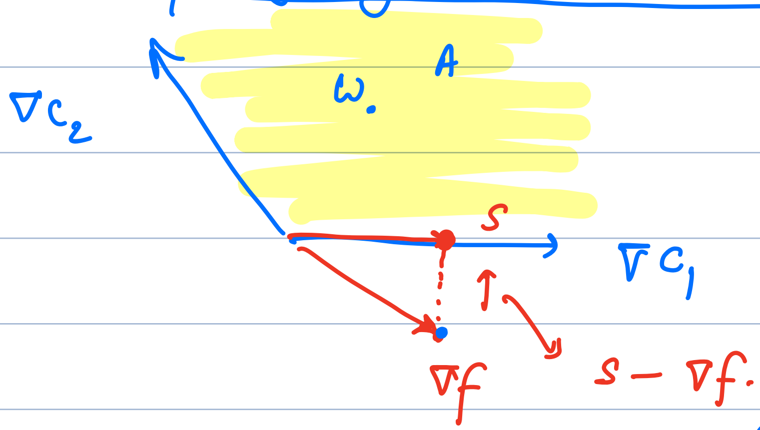


$\rightarrow \nabla c_1^T d \geq 0$ and

$\rightarrow \nabla c_2^T d \geq 0$.

↳ Need to show $\nabla f \in A$.

<u>proof by contradiction:</u> let $\nabla f \notin A$



$$s = \underset{w \in A}{\operatorname{argmin}} \; \| w - \nabla f \|$$

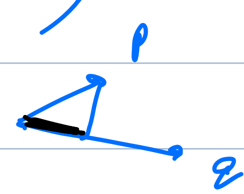
<u>Claim:</u> $s - \nabla f$ is both

↳ feasible direction &

↳ descent direction.

If this is true ⇒ x can be improved by moving along $s - \nabla f$. , contradicts x^* being local minima

↳ Arguments for this claim: $(p^T q)$

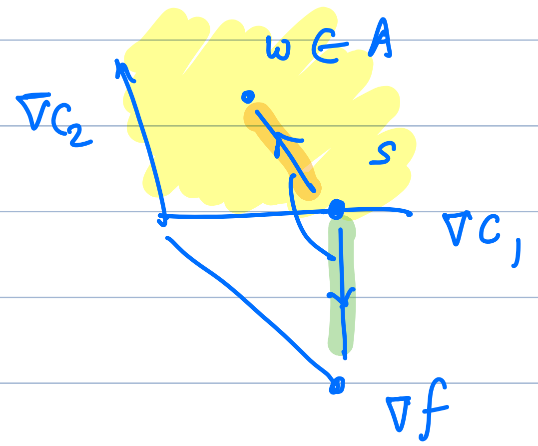


① $s^T \nabla f < \nabla f^T \nabla f$

$$\Rightarrow (s - \nabla f)^T \nabla f < 0$$

$\Rightarrow$ $(s - \nabla f)$ is a descent direction.

②



$$(w - s)^T (\nabla f - s) \leq 0$$

by property of convex sets

Take $w = s + \nabla c_1$

$$\Rightarrow \quad \nabla c_1^T (\nabla f - s) \leq 0$$

$$\Rightarrow \quad \nabla c_1^T (s - \nabla f) \geq 0$$

Similarly $w = s + \nabla c_2$

$$\Rightarrow \quad \nabla c_2^T (s - \nabla f) \geq 0$$

$\Rightarrow$ $(s - \nabla f)$ is a feasible dir.

$\Rightarrow$ x can be improved!

$\Rightarrow$ a contradiction

$\Rightarrow$ $\underline{\underline{\nabla f \in A}}$

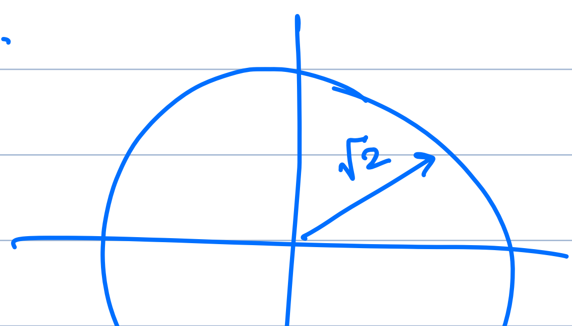
Rest follows.

——— x ———

$f(x) = x_1 + x_2.$

Some simple verification.

1) $\underline{x_A = (-1, 0)}$



$\nabla f = (1$

$$\nabla c_1 = -2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad c_1 = 2 - x_1^2 - x_2^2 \geq 0$$

$$\nabla c_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad c_2 = x_2 \geq 0$$

$c_1(x_A) = 1 > 0 \Rightarrow$ inactive

$c_2(x_A) = 0$ active $\qquad \nabla c_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Is x_A a optimal point :

Is there a λ_2 s.t. $\nabla f = \lambda_2 \nabla c_2$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\rightarrow$ NO λ_2 works.

$\therefore x_A$ doesn't satisfy KKT.

2) $x_B = (-\sqrt{2}, 0)$

$\left.\begin{aligned} c_1(x) &= 0 \\ c_2(x) &= 0 \end{aligned}\right\} \Rightarrow$ Is there a λ_1, λ_2 s.t.

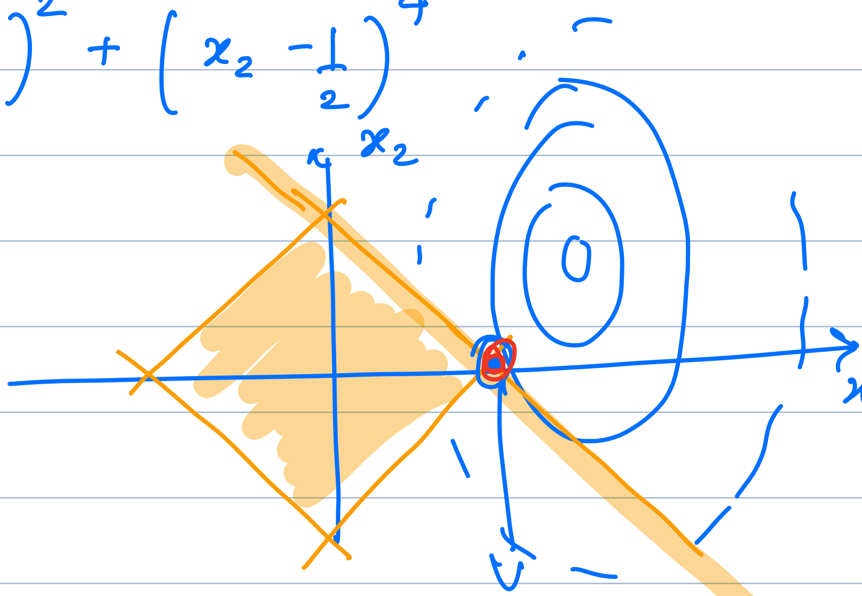
$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

YES, $\lambda_1 = \frac{1}{2\sqrt{2}}, \ \lambda_2 = 1.$

$\therefore$ does satisfy KKT. condn.

e.g. $\min_{x} \left(x_1 - \frac{3}{2} \right)^2 + \left(x_2 - \frac{1}{2} \right)^4$

s.t. $\begin{bmatrix} 1 - x_1 - x_2 \\ 1 - x_1 + x_2 \\ 1 + x_1 - x_2 \\ 1 + x_1 + x_2 \end{bmatrix} \geq 0$



Which constraints are active at True soln $\rightarrow (1,0)$

c_1 ✓, c_2 ✓ active

$$\nabla_x \mathcal{L} = 0 = \nabla f - \lambda_1 \nabla c_1 - \lambda_2 \nabla c_2$$

$(1,0)$ $\quad \nabla f \rightarrow \begin{pmatrix} 2(x_1 - 3/2) \\ 4(x_2 - 1/2)^3 \end{pmatrix}, \quad \nabla c_1 \rightarrow \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad \nabla c_2 \rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} -1 \\ -1/2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

$$\underbrace{\underbrace{\lambda_1 = \frac{3}{4}, \ \lambda_2 = \frac{1}{4}}_{\geq 0} \ ✓, \ \lambda_3 = 0, \ \lambda_4 = 0}$$