↳ **Matrix Factorizations**

1) LU decomposition

$$A = LU \longrightarrow PA = LU$$

P ↓ permutation matrix

want to solve $Ax = b$.

$$L\underbrace{Ux}_{y} = b \quad , \qquad Ly = b$$

$$\begin{bmatrix} \diagdown & 0 \\ & \diagdown \end{bmatrix} y = \begin{bmatrix} b \end{bmatrix}$$

$$Ux = y$$

$$\begin{bmatrix} \diagdown & \\ & \diagdown \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = y \qquad L_{11} y_1 = b_1$$

$$U_{NN} x_N = y_N$$

↳ A is sym pos def: Cholesky decomp:

$$A = LL^T$$

↳ Rectangular → QR, $A = QR$ → upper triangular.

Q ↓ Orthogonal

— x —

Analysis

↳ Sequences. $\{x_k\}_{k=1}^{\infty}$

→ Seq converges to some pt x^* $\lim_{k \to \infty} x_k = x^*$

→ if for any $\varepsilon > 0$, there is an index K such that $\| x_k - x^* \| \leq \varepsilon \quad \forall\, k > K$.

↳ Subsequence : index set $S \subset \{1, 2, 3, \ldots\}$

↳ $\{x_k\}_{k \in S}$. Accumulation / limit pt $\hat{x}$

eg. $1, \frac{1}{3}, 2, 1, \frac{1}{9}, 2, 1, \frac{1}{81}, 2, \ldots$

$\{x_1, x_4, x_7 \ldots\} \longrightarrow 1$

$\{x_2, x_5, x_8 \ldots\} \longrightarrow 0$

$\{x_3, x_6, \ldots\} \longrightarrow 2$

A sequence converges iff it has exactly one limit pt.

↳ Cauchy sequence : if for $\varepsilon \geq 0$ there exists an integer K s.t $\| x_k - x_l \| \leq \varepsilon \quad \forall\, k \geq K$ and $l \geq K$. A seq converges iff it is a Cauchy seq.

→ bounded above $t_k \leq u \quad \forall\, k$

→ nondecreasing $t_{k+1} \geq t_k \quad \forall\, k$

Rates of Convergence

$\{x_k\} \xrightarrow{\text{Conv}} x^*$ "Q" → quotient

Q-linear | Q-superlinear | Q-quadratic

1) $\dfrac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} \leq r$ for large k & $r \in (0,1)$

2) $\lim\limits_{k\to\infty} \dfrac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|} = 0$

3) $\dfrac{\|x_{k+1} - x^*\|}{\|x_k - x^*\|^2} \leq M$ where $M > 0$

↪ The seq $x_k = 1/k!$?

1) converges to 0

2) $\dfrac{|x_{k+1}|}{|x_k|} = \dfrac{1}{k+1}$. As $k \to \infty$

(a) $\to 0$

Q-superlinear.

3) Is it Q-quadratic? $\dfrac{|x_{k+1}|}{|x_k|^2} = \dfrac{k!}{(k+1)} \to \infty$ as $k \to \infty$

↪ Topology set F

eg. bounded set $\|x\| \leq M \;\forall x \in F$

↳ Convex Sets : interesting combinations.

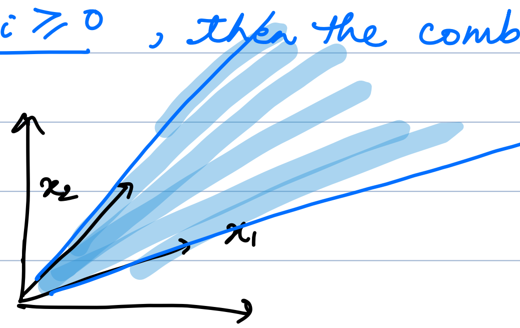
$x_1, x_2, \ldots, x_k$

$\alpha_i \in \mathbb{R}$

1) $y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k$

'Linear combination.

2) if we say $\alpha_i \geq 0$, then the combination is conic



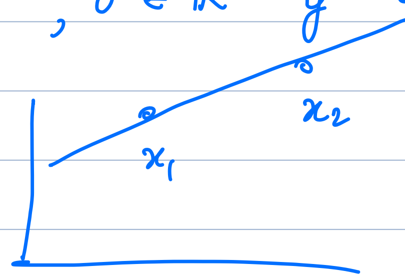
3) if $\sum \alpha_i = 1$

is called affine combination.

4) if conic + affine, ie. $\alpha_i \geq 0$ & $\sum \alpha_i = 1$

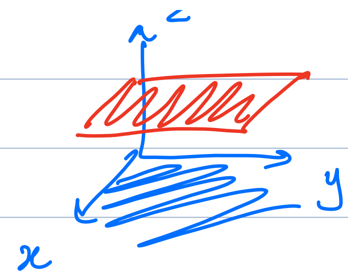
Convex combination.

→ $x_1, x_2 \in \mathbb{R}^n$, $\theta \in \mathbb{R}$ $y = \theta x_1 + (1-\theta) x_2$



↳ Affine Set $S \subset \mathbb{R}^n$ is affine if the line through any two distinct points in S lies in S

e.g.



say we have a v.s. V

$x \in V$, $b \notin V$

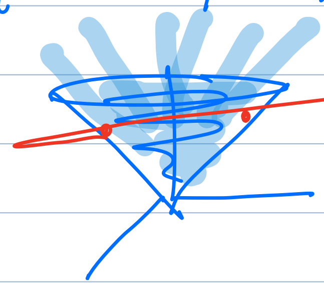
Affine set $S: \{ y = x + b \mid x \in V, b \notin V \}$

$$\left.\begin{matrix} y_1 \in S \\ y_2 \in S \end{matrix}\right\} \rightarrow \theta y_1 + (1-\theta) y_2 = \underbrace{\theta x_1 + (1-\theta) x_2}_{x' \in V} + b$$

$$\Rightarrow \quad x' + b \quad \therefore \in S. \quad \checkmark$$

↳ **Affine Hull** : The set of all affine combinations of points in some set S.

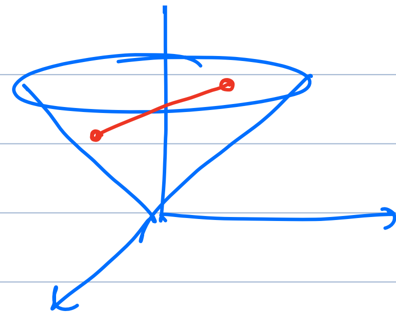
$$S: \{ x \in \mathbb{R}^3 \mid x_3 \geq \sqrt{x_1^2 + x_2^2} \}$$



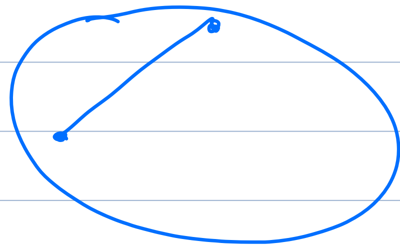
aff S ? $\mathbb{R}^3$

↳ **Convex Set** : S if the line segment between any 2 pts in S lies in S.

$x_1, x_2 \in S$, $\quad \theta x_1 + (1-\theta) x_2 \in S$, $\quad \underbrace{0 \leq \theta \leq 1}$.

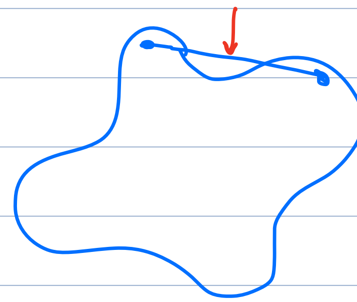


Convex Hull: Set of all convex combinations of pts in the set.



Convex

(a)

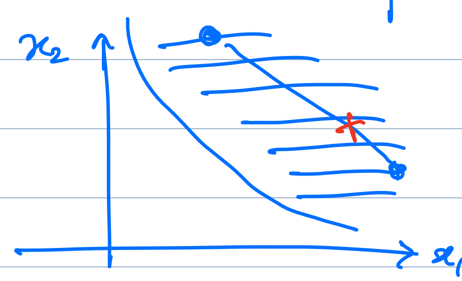


Not convex

(b)

↳ Define hyperbolic set: $S = \{ x \in \mathbb{R}^2_+ \mid x_1 x_2 \geq 1 \}$

Is it convex?



$$\left\{ \begin{array}{lll} Pt_1 = (x_1, x_2) & \rightarrow & x_1 x_2 \geq 1 \\ P_2 = (y_1, y_2) & \rightarrow & y_1 y_2 \geq 1 \end{array} \right.$$

c.c

$$P = (\alpha x_1 + \beta y_1, \ \alpha x_2 + \beta y_2) \qquad \text{s.t } \alpha + \beta = 1, \ \alpha, \beta \geq 0$$

$$(\alpha x_1 + \beta y_1)(\alpha x_2 + \beta y_2)$$

$$= \alpha^2 \underbrace{x_1 x_2}_{\geq} + \beta^2 \underbrace{y_1 y_2}_{\geq 1} + \alpha\beta \underbrace{(x_1 y_2 + y_1 x_2)}_{\geq 2 \text{ by AM-GM}}$$

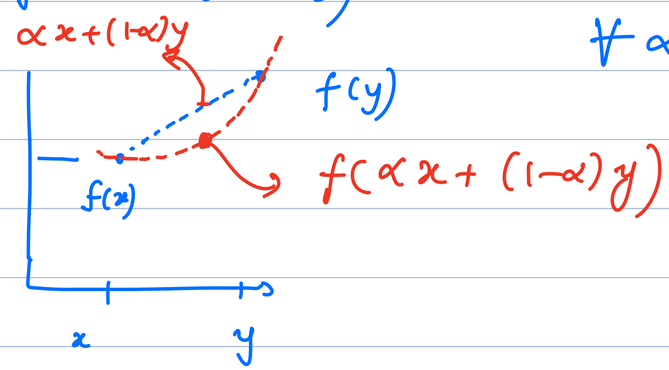
$\geq (\alpha+\beta)^2 = 1$

↳ <u>Convex functions</u> : A fn f is a convex fn if

1) domain is a convex set
2) given any 2 pts in the domain,

→ $f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y)$

$\forall \alpha \in [0,1]$



ex. $f(x) = a^T x + b$, $x^T H x$ H is sym pos def.

e.g. $f(x) = x^T A x + b^T x + c$, A is sym pos. semi def.

show convexity of f ?

→ $x, y, \frac{x+y}{2}$: verify $\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)$

$\frac{1}{2}\left(x^T A x + b^T x + c\right) + \frac{1}{2}\left(y^T A y + b^T y + c\right)$

$- \left(\frac{x+y}{2}\right)^T A \left(\frac{x+y}{2}\right) + b^T\left(\frac{x+y}{2}\right) + c$

$$= \frac{1}{2} (x-y)^T A (x-y) \geq 0$$

because A is pos. sem. def.

$x^T A x > 0 \quad \forall x \quad A:$ sym pos. def

$x^T A x \geq 0 \quad \forall x \quad A:$ sym semi pos def.

MM