# EE5121 : Summary of Background

Lin Alg + Analysis

## Linear Algebra

1. $x$ in n-dimspace $\mathbb{R}^n$ (real) & $\mathbb{C}^n$ (complex)

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad x^T = [\quad\quad], \quad x^H = [x_1^* \;\; x_2^* \;\; \cdots \;\; x_n^*]$$

2) $A \in \mathbb{R}^{m\times n}$ m: rows, n: cols.

element of $A = A_{ij} \neq a_{ij}$

→ Positive definite matrix : $x^T A x > 0, \; \forall x \in \mathbb{R}^n$ $\geq$

e.g. Gram matrix : set of vectors $\{v_i\}_{i=1}^n$

$G_{ij} = \langle v_i, v_j \rangle$. Positive semi definite?

arrange $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \\ | & & & | \end{bmatrix}$

$G = A^T A$. Consider $x^T G x = x^T A^T A x$

$= \|Ax\|_2^2 \geq 0$

$$\begin{bmatrix} v_1 & v_2 & & v_N \\ \vdots & \vdots & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_N \end{bmatrix} = \sum v_i x_i$$

$(\sum_i v_i x_i)^2 \geq 0$ depending on $\{v_i\}$ being linearly independent.

# Norms

→ vector norm: $\|x\|_2 = \left(\sum |x_i|^2\right)^{1/2}$

↳ p-norms:

$$\|x\|_p = \left(\sum |x_i|^p\right)^{1/p}, \quad 1 \leq p \leq \infty$$

$$\|x\|_\infty = \max |x_i|$$

→ Properties

a) $\|x\| = 0 \implies x = 0$

b) $\|\alpha x\| = |\alpha| \, \|x\|, \quad \alpha \in \mathbb{R}$

c) $\|x + z\| \leq \|x\| + \|z\|$

Proof: Minkowski's inequality

↳ why is $p < 1$ not a norm?

$p = 1/2$. $\quad x = (1, 0), \quad z = (0, 1)$

$\|x\| = \|z\| = 1$

$\|x + z\| = 4$

↳ dual norm: $\|x\|_D = \max\limits_{\|y\| = 1} x^T y$

→ Matrix norms:

↳ norms that are consistent with the vector norms: $\|A\|_1 = \sup\limits_{x \neq 0} \dfrac{\|Ax\|_1}{\|x\|_1}$. ←

↳ eg: $\|A\|_2 = \text{largest eig}(A^T A)^{1/2}$

$$\|y\|_1 = \sum_i |y_i|$$

↳ Take the case of $\|A\|_1$ ?

Max col sum of the absolute values of $A$.

→ $\|Ax\|_1 = \sum_i \left| \sum_j A_{ij} x_j \right|$

Triangle ineq: $\left| \sum_j A_{ij} x_j \right| \leq \sum_j |A_{ij}||x_j|$

$\Rightarrow \|Ax\|_1 \leq \sum_i \sum_j |A_{ij}||x_j| = \sum_j |x_j| \underbrace{\sum_i |A_{ij}|}_{S_j}$

$\leq \sum_j |x_j| S_j \leq S_m \sum_j |x_j|$

$S_m$ → max value of $\{S_j\}$.

defn $\|A\|_1 \leq S_m$

↳ $\|A\|_1 = \sup_{x \neq 0} \frac{\|Ax\|_1}{\|x\|}$

$x = [0 \ldots 1 \quad 0]$ . $Ax = A_m$

$1$ → $m^{th}$ index $\qquad \|x\|_1 = 1$

$\therefore \|A\|_1 = S_m$.

$\|A_m\|_1 = S_m$

↳ Non consistent norms: Frobenius norm:

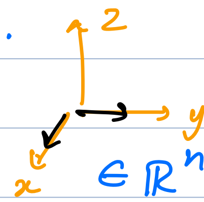
$$\|A\|_F = \left( \sum_i \sum_j |A_{ij}|^2 \right)^{1/2}$$

↳ Condition number. Non singular matrix,

$$\kappa(A) = \|A\| \times \|A^{-1}\|.$$

# Subspaces (Vector spaces)

- The subset $S \subset \mathbb{R}^n$ is a subspace : if given any 2 elements in it, the linear combination also lies in it.




- Set of vectors $\{s_1, \dots s_m\} \in \mathbb{R}^n$
  - → linearly independent
  - → spanning set

- Combine the above 2 : The set above is a basis if a) they are linearly indepn & b) they are a spanning set.

- $$\overset{m \times n}{A}\ \underset{n \times 1}{x} = y \rightarrow m \times 1$$

| $\mathbb{R}^n$ | $\mathbb{R}^m$ |
|---|---|
| Row space of $A$ (dim $r$) | Col space of $A$ / Range $(A)$ (dim $r$) |
| $\{x \mid x = A^T y,\ y \in \mathbb{R}^m\}$ | $\{y \mid y = Ax,\ x \in \mathbb{R}^n\}$ |
| Null space of $A$ dim $(n-r)$ | Left null space dim $(m-r)$ |
| $\{x \mid Ax = 0\}$ | $\{y \mid A^T y = 0\}$ $\Leftrightarrow$ $y^T A = 0$ |

F.T.L.A → Row & Null spaces :

→ orthogonal to each other

→ direct sum is $\mathbb{R}^n$

Col & L Null space

→ orthogonal

→ direct sum is $\mathbb{R}^m$

↳ Eigen and SVD analysis.

↳ $Ax = \lambda x$

sq matrix ↳ scalar, eig value

↳ If A is symmetric → eigvalues are all real.

Spectral decomposition : $A = Q \Lambda Q^T$ ↳ diagonal matrix

$= \sum_i \lambda_i q_i q_i^T$ → eigvectors of A.

↳ If in addn it is positive definite → eigvalues are positive real nos.

$x^T A x > 0 \quad \forall x$

$\Rightarrow \sum_i x^T \lambda_i q_i q_i^T x = \sum_i \lambda_i \underbrace{x^T q_i} \underbrace{q_i^T x}$

$= \sum_i \lambda_i \underbrace{(q_i^T x)}^2 > 0$

possible $\forall x$ if $\lambda_i > 0 \ \forall i$.

$\begin{bmatrix} \sigma_1 & \\ \end{bmatrix}$

↳ SVD, $A = U \Sigma V^T$ $\qquad S = \begin{bmatrix} \ddots & \\ & \sigma_r \end{bmatrix}$

$U, V$ are square, orthogonal.

$\Sigma \to$ 1. $A$ is a tall matrix, $\Sigma = \begin{bmatrix} S \\ 0 \end{bmatrix}$

2. $A$ is a fat matrix, $\Sigma = [S \;\; 0]$

3. $A$ is square, $\Sigma = [S]$

Convention: $\sigma_1 \to$ largest $\qquad \sigma_i \geq 0$

$\sigma_n \to$ smallest

$$\to A = \sum_i \sigma_i u_i v_i^T$$

↳ "pseudo inverse" of a matrix $A^+ = V \Sigma^+ U^T$

$\Sigma^+ \to$ same size as $\Sigma^T$

$\Sigma^+_{ii} = 1/\sigma_i$

$Ax = b$

$x = A^+ b$ under some condns.

↳ $\|A\|_2 = \sigma_1(A)$, $\|A^{-1}\| = \dfrac{1}{\sigma_n(A)}$

$$\Rightarrow K = \frac{\sigma_1}{\sigma_n}$$

Impact of condn no:

$Ax = b \longleftrightarrow \hat{A}\hat{x} = \hat{b}$

$$\frac{\|x-\hat{x}\|}{\|x\|} = \boxed{K(A)} \left[ \frac{\|A-\hat{A}\|}{\|A\|} + \frac{\|b-\hat{b}\|}{\|b\|} \right]$$

e.g. $A = \hat{A}$, % change in $b$ = 5 %

$K(A) = 10$

$\Rightarrow$ % change in $x$ = 50 %

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad \hat{b} = \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \end{bmatrix}, \quad \|b-\hat{b}\|^2 = \left[ (b_1-\hat{b}_1)^2 + (b_2-\hat{b}_2)^2 \right]$$