EE2025 Engineering Electromagnetics: July-Nov 2019
Tutorial 7: Antennas and Antenna Arrays

1. (a) Find the electric and magnetic fields generated by an Hertz dipole antenna?
   (b) Show that the power radiated in a far field and near field regions from a Hertzian dipole is independent of the distance of a test object from the antenna.

Solution:

(a) (Derivation is available in section 8.2 of Shevagaonkar's textbook)

1. Specify $\overrightarrow{\mathbf{J}}$ (electric current density source)

$$\overrightarrow{\mathbf{I}}(x', y', z') = I_0 e^{jwt}\hat{a_z} \quad when\, x = 0,\, y = 0,\, dl/2 \leq z' \leq dl/2$$

Where $\overrightarrow{\mathbf{I}}(x', y', z') = \int_x \int_y \overrightarrow{\mathbf{J}}\, dxdy$

2. find $\overrightarrow{\mathbf{A}}$

$$\begin{aligned}
\overrightarrow{\mathbf{A}} &= \frac{\mu}{4\pi}\iiint_V \overrightarrow{\mathbf{J}}\frac{e^{-j\beta r}}{r}dv' \\
&= \frac{\mu}{4\pi}\int_{-dl/2}^{dl/2} I_0 e^{jwt}\hat{a_z}\frac{e^{-j\beta r}}{r}dz' \\
&= \frac{\mu}{4\pi}I_0 dl\frac{e^{-j\beta r}}{r}e^{jwt}\hat{a_z} \quad (\text{Assuming}\, dl \lll r) \\
&= A_z\hat{a_z} = A_z\cos\theta\hat{a_r} - A_z\sin\theta\hat{a_\theta} + 0\hat{a_\phi}\,(\text{spherical coordinates})
\end{aligned}$$

where $k^2 = w^2\mu\epsilon$, $r = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$.

3. Find the $\overrightarrow{\mathbf{E}}$, $\overrightarrow{\mathbf{H}}$ fields using

$$\overrightarrow{\mathbf{H}} = \frac{1}{\mu}\nabla \times \overrightarrow{\mathbf{A}}$$

$$\overrightarrow{\mathbf{E}} = \frac{1}{j\omega\epsilon}\nabla \times \overrightarrow{\mathbf{H}} = -j\omega\overrightarrow{\mathbf{A}} - \frac{j}{\omega\mu\epsilon}\nabla(\nabla.\overrightarrow{\mathbf{A}})$$

(b) The expressions for **E** and **H** field components for a Hertzian dipole in a spherical coordinate system are:

$$E_r = \frac{I_0 dl}{2\pi}\eta\left[\frac{1}{r} - \frac{j}{kr^2}\right]\frac{e^{-jkr}}{r}\cos\theta;$$

$$E_\theta = \frac{I_0 dl j\omega\mu}{4\pi}\left[1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right]\frac{e^{-jkr}}{r}\sin\theta; and\ E_\phi = 0$$

$$H_\phi = \frac{I_0 dl}{4\pi}jk\left[1 + \frac{1}{jkr}\right]\frac{e^{-jkr}}{r}\sin\theta;\ and\ H_r = H_\theta = 0$$

The time-average power density (S_{av}):

$$\mathbf{S_{av}} = \frac{1}{2}Re[\mathbf{E}X\mathbf{H}^*] = \frac{1}{2}Re[E_\theta H_\phi^*\hat{r} - E_r H_\phi^*\hat{\theta}]$$

Now we surround the dipole with an imaginary sphere of radius r and compute the radiated power W by taking the surface integral of the power density $\mathbf{S_{av}}$: i.e.

$$W = \int_S \mathbf{S_{av}}\cdot\mathbf{ds} = \frac{1}{2}Re\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}[E_\theta H_\phi^*\hat{r} - E_r H_\phi^*\hat{\theta}]\cdot r^2\sin\theta d\theta d\phi\ \hat{r}$$

$$\implies W = \frac{1}{2} Re \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} E_\theta H_\phi^* r^2 \sin\theta d\theta d\phi$$

$$W = \frac{1}{2} Re \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\eta(I_0 dl)^2}{16\pi^2} sin^2\theta \left[\frac{jk}{r} + \frac{1}{r^2} - \frac{j}{kr^3}\right]\left[-\frac{jk}{r} + \frac{1}{r^2}\right] r^2 \sin\theta d\theta d\phi$$

Considering only the real parts, and rearranging result in;

$$W = \frac{1}{2} \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\eta(I_0 dl)^2}{16\pi^2} sin^3\theta \left[\frac{k^2}{r^2} + \frac{1}{r^4} - \frac{k}{kr^4}\right] r^2 d\theta d\phi$$

$$W = \frac{1}{2} \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\eta k^2 (I_0 dl)^2}{16\pi^2} sin^3\theta d\theta d\phi$$

$$W = \frac{1}{2} \frac{\eta k^2 (I_0 dl)^2}{16\pi^2} 2\pi \int_{\theta=0}^{\pi} sin^3\theta d\theta$$

Upon substituting, $k = \frac{2\pi}{\lambda}$ and $\int_{\theta=0}^{\pi} sin^3\theta d\theta = 4/3$;

$$W = \frac{I_0^2 \pi \eta}{3}\left(\frac{dl}{\lambda}\right)^2$$

W is independent of the term r in the above expression. Thus, the radiated power in near field or in far field conditions is independent of the radial distance r.

2. A $1m$ long dipole is excited by a $5MHz$ current with an amplitude of $5A$. At a distance of $2km$, what is the power density radiated by the antenna along $\theta = 90°$ (broadside direction)?

Solution:

$$S = \frac{\eta_0 k^2 I^2 l^2}{32\pi^2 R^2} \sin^2\theta \tag{1}$$

We know that $\eta_0 = 120\pi$, $\lambda = 60m$, $l = 1m$, $R = 2000m$, $I = 5A$, $k = 2\pi/\lambda$. Substituting these values, we get $S = 8.18 \times 10^{-8} W/m^2$

3. The power radiated by a lossless antenna is 10 watts. The directional characteristics of the antenna is represented by the radiation intensity of $U(\theta, \phi) = B_0 \cos^3\theta$ (W/unit solid angle) $0 \le \theta \le \pi/2$ $0 \le \phi \le 2\pi$ Find
a) Maximum power density (in watts per square meter) at a distance of 1000m (assume far field). Specify the angle where it occurs.
b) Directivity of the antenna (dimensionless and in dB)

Solution: a) Average power is given by
$S_{avg} = \int_0^{2\pi}\int_0^{\pi/2} U(\theta,\phi) sin\theta d\theta d\phi = B_0 \int_0^{2\pi}\int_0^{\pi/2} cos^3\theta sin\theta d\theta$
$10 = 2\pi B_0 \int_0^{\pi/2} cos^3\theta sin\theta d\theta d\phi$
$B_0 = 6.36$
$U(\theta,\phi) = 6.36 cos^3\theta$
$W = \frac{U}{r^2} = \frac{6.36 cos^3\theta}{r^2}$
$W_{max} = \frac{U_{max}}{r^2} = \frac{6.36}{r^2} = 6.36 \times 10^{-6} W/m^2$
Angle at which we have maximum power density is $\theta = 0$

b) $D_0 = \frac{4\pi U_{max}}{S_{avg}} = 8 = 9\text{dB}$

4. Calculate the directivity, total power radiated and radiation resistance of an half-wave dipole (length $= \lambda/2$) antenna with far fields given below:

$$E_\theta \simeq \frac{j\eta I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right]$$

$$H_\phi \simeq \frac{jI_0 e^{-jkr}}{2\pi r}\left[\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right]$$

Hint: $\int_0^\pi \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin\theta}d\theta = 1.21883$

Solution: Time-average power density:

$$\begin{aligned}
\overrightarrow{S_{av}} &= \frac{1}{2}Re(\overrightarrow{\mathbf{E}} \times \overrightarrow{\mathbf{H}}^*) \\
&= \frac{1}{2}Re\left[\frac{j\eta I_0 e^{-jkr}}{2\pi r}\left(\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right)\hat{a_\theta} \times \frac{-jI_0 e^{jkr}}{2\pi r}\left(\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right)\hat{a_\phi}\right] \\
&= \frac{\eta}{2}\left[\frac{I_0}{2\pi r}\left(\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right)\right]^2 \hat{a_r}
\end{aligned}$$

Total power radiated:

$$\begin{aligned}
W &= \iint_S \overrightarrow{S_{av}}.\overrightarrow{ds} = \int_0^{2\pi}\int_0^\pi \overrightarrow{S_{av}}.(r^2\sin\theta d\theta d\phi \hat{a_r}) \\
&= \int_0^{2\pi}\int_0^\pi \frac{\eta I_0^2}{8\pi^2}\left(\frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right)d\theta d\phi \\
&= \left[\int_0^{2\pi} d\phi\right]\left[\int_0^\pi \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin\theta}d\theta\right]\frac{\eta I_0^2}{8\pi^2} \\
&= \left[2\pi\right]\left[1.21883\right]\frac{\eta I_0^2}{8\pi^2} = \frac{1.21883\eta I_0^2}{4\pi}
\end{aligned}$$

Radiation intensity:

$$U(\theta,\phi) = r^2|\overrightarrow{S_{av}}| = \frac{\eta}{2}\left[\frac{I_0}{2\pi}\left(\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right)\right]^2$$

$$U_{max} = \frac{\eta}{2}\left[\frac{I_0}{2\pi}\right]^2$$

Directivity:

$$D(\theta,\phi) = 4\pi\frac{U(\theta,\phi)}{W} = 1.641\left(\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right)^2$$

(Maximum Directivity = 1.641.)
Radiation Resistance(R_r):

$$\begin{aligned}
W &= \frac{I_0^2 R_r}{2} = \frac{1.21883\eta I_0^2}{4\pi} \\
\implies R_r &= \frac{1.21883\eta}{2\pi} = \frac{1.21883 \times 120\pi}{2\pi} = 73.13\Omega
\end{aligned}$$

Half-wave dipoles and mono-poles are the most commonly used antennas. Monopole antennas have only one pole with length $\lambda/4$ and the other pole is replaced with ground plate. Monopole antenna has a radiation resistance of $73.12/2\Omega = 36.56\Omega$ (half of half-wave dipoles as they radiate only to one hemisphere).

5. An antenna has been designed as a half wavelength dipole for use with a TV transmitter at 600 MHz (UHF channel 35) and the transmitter supplies 50 kW to the antenna. The antenna is 6 mm thick and made of aluminium with conductivity $\sigma = 3 \times 10^7$ S/m. Consider the radiation resistance of dipole (R_{rad}) is 73.08 Ω and the current is uniform. Calculate:
(a) The radiated power at 600 MHz
(b) The efficiency of the antenna ($eff = \frac{P_{rad}}{P_{in}} = \frac{R_{rad}}{R_{rad}+R_{in}}$)

Solution: The length of the antenna is

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{600 \times 10^6} = 0.5 \Longrightarrow L = 0.25$$

To find the radiated power, we need to find the internal resistance and radiation resistance of the antenna. The radiation resistance (R_{rad}) of the half wave dipole is 73.08 Ω. Internal resistance can be found by using the equation for resistance of a hollow cylinder

$$R_d = \frac{L}{\sigma S}$$

Where S is,

$$S = 2\pi r\delta$$

where δ is the skin depth. In this problem, diameter is given as 6 mm (r=3 mm).

$$\delta = \frac{1}{\sqrt{\pi f \mu_0 \sigma}} = 3.751 \times 10^{-6} m$$

$$R_d = \frac{L}{2\pi r\delta\sigma} = 0.1179\,\Omega$$

$$P_{in} = \frac{I^2(R_{rad} + R_d)}{2} \Longrightarrow I^2 = 1366.16 \Longrightarrow I = 36.96\,A$$

Radiated power

$$P_{rad} = \frac{I^2 R_{rad}}{2} = 49.919\,KW$$

(b) The efficiency of the antenna

$$eff = \frac{P_{rad}}{P_{in}} = 99.84\,\%$$

This is a very high efficiency beacause of its low internal resistance.

6. Four isotropic sources are placed along the z-axis as shown in Fig.1. Assuming that excitations of elements (current fed to the elements) 1 and 2 are +1 and the excitations of elements 3 and 4 are −1 (180^0 out of phase with 1 and 2). Find
(a) The array factor in simplified form
(b) All the nulls when $d = \frac{\lambda}{2}$.

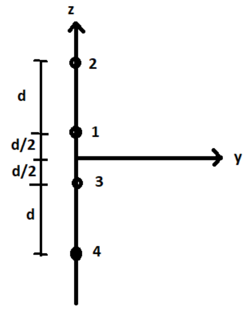


Figure 1: isotropic sources

Solution: (a)

$$E = K\left[\frac{e^{-jkr_2}}{r_2} + \frac{e^{-jkr_1}}{r_1} - \frac{e^{-jkr_3}}{r_3} - \frac{e^{-jkr_4}}{r_4}\right]$$

where,

$$r_1 = r - \frac{d}{2}cos(\theta)$$

$$r_2 = r - \frac{3d}{2}cos(\theta)$$

$$r_3 = r + \frac{d}{2}cos(\theta)$$

$$r_4 = r + \frac{3d}{2}cos(\theta)$$

K is a constant.

$$E = \frac{Ke^{-jkr}}{r}\left[e^{\frac{j3kd}{2}cos(\theta)} + e^{\frac{jkd}{2}cos(\theta)} - e^{-\frac{jkd}{2}cos(\theta)} - e^{-\frac{j3kd}{2}cos(\theta)}\right]$$

For amplitude variations, $r_1 \approx r_2 \approx r_3 \approx r_4 \approx r$

$$AF = 2j[sin(\frac{3kd}{2}cos\theta) + sin(\frac{kd}{2}cos\theta)]$$

Let $x = kdcos\theta$, $y = \frac{kd}{2}cos\theta$

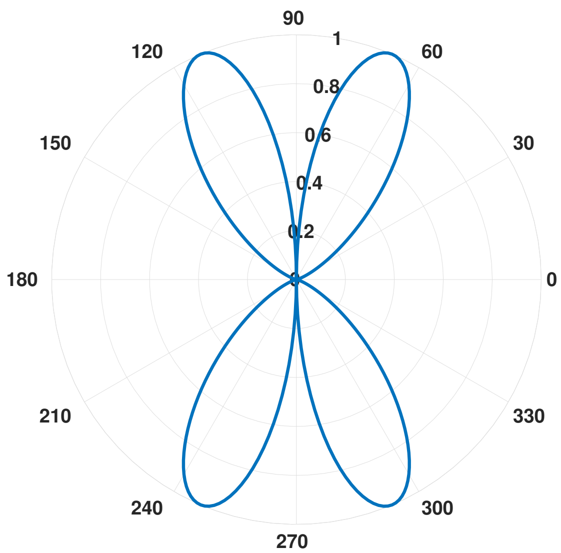
$$AF = 4j[sin(kdcos\theta)cos(\frac{kd}{2}cos\theta)]$$

(b)

$$AF(d = \frac{\lambda}{2}) = 4j[sin(\pi cos\theta)cos(\frac{\pi}{2}cos\theta)]$$

The nulls will be placed at θ such that $AF(\theta) = 0$.

$$\theta_n = 0, 90, 180$$



(Prajosh)

7. Consider two isotropic antennas kept in a plane separated by a distance of two wavelengths. If both the antennas are fed with currents of equal phase and magnitude. Calculate the number of lobes in the radiation pattern.

Solution: Magnitude and phase of currents in both the antennas are same and d, distance between the two antennas is 2λ.Hence the electric field can be written as:

$$E = 1 + 1e^{\beta d\cos\theta}$$

Solving the above equation,we have $E = 2e^{\frac{j\beta d\cos\theta}{2}}(\frac{e^{\frac{j\beta d\cos\theta}{2}} + e^{\frac{-j\beta d\cos\theta}{2}}}{2})$

$$E = 2\cos\frac{\psi}{2}e^{\frac{j\beta d\cos\theta}{2}}$$

$$\psi = \beta d\cos\theta = 4\pi\cos\theta$$

Hence the magnitude of electric field comes out to be

$$E = 2\cos(\frac{\psi}{2}) = 2\cos(2\pi\cos\theta)$$

As θ varies from 0 to 2π.Maximum can occur at

$$\theta = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$$

Hence there are 8 lobes in the radiation pattern.

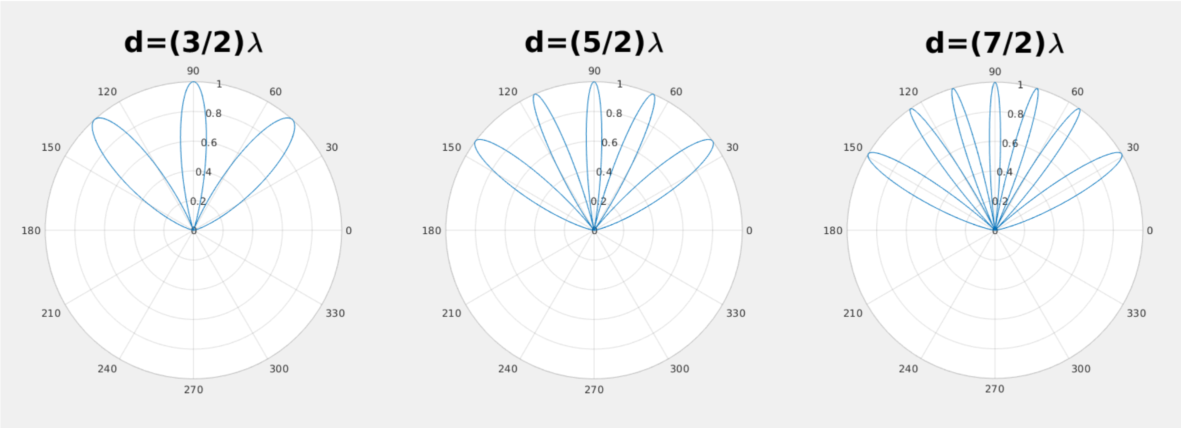
8. Obtain an expression for the array factor of a two-element array of isotropic antennas with equal excitation and a separation $d = n\lambda/2$ where $n \in \mathbb{N}$. The array is along the x-axis. How many beams are there between $\phi = 0$ and $\phi = 180$ in the radiation pattern? Do they have the same width and peak value?

Solution: Since the two antennas have equal excitation, $a_0 = a_1 = 1$. The array factor expression simplifies to

$$AF = |1 + e^{j\gamma}| = |2\cos^2(\gamma/2) + 2j\sin(\gamma/2)\cos(\gamma/2)| = 2\cos(\gamma/2)$$

where $\gamma = (2\pi d/\lambda)\cos\theta$.

Given $d = n\lambda/2$. Since $AF = 2\cos(\pi(n/2)\cos\theta)$, for $\phi \in (0, \pi)$, radiation pattern has n beams. They will have the same peak, but different widths. Given below are the array factors for $d = \{1.5\lambda, 2.5\lambda, 3.5\lambda\}$. As expected, there are 3, 5 and 7 beams respectively.



9. For a uniform linear array of N isotropic antennas, determine the directivity of the antennas when the spacing between the elements is 'd' and further find directivity when d is a) $\lambda/4$
b) $\lambda/2$

Solution: The maximum directivity (which occurs in the broadside direction) of a uniformly excited equally spaced linear array can be found as follows. First, the normalizedarray factor is

$$(AF)_n = \frac{\sin(\frac{N}{2}kd\cos\theta)}{N\sin(\frac{1}{2}kd\cos\theta)} \tag{2}$$

If $d << \lambda$, we can employ the small angle approximation for the denominator, yielding $D_0 = 2N(d/\lambda)$

$$(AF)_n = \frac{\sin(\frac{N}{2}kd\cos\theta)}{\frac{N}{2}kd\cos\theta} \tag{3}$$

The average radiation intensity is found using

$$U_0 = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left[\frac{\sin(\frac{N}{2}kd\cos\theta)}{\frac{N}{2}kd\cos\theta}\right]\sin\theta d\theta d\phi \tag{4}$$

Now, this integral is not easy to evaluate there are assumptions made on N which can approximate U_0. The directivity is

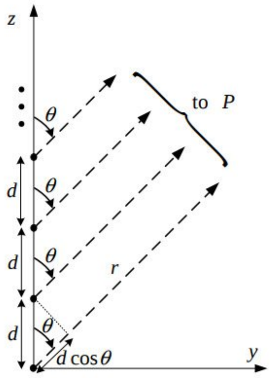
$$D = 2\frac{Nd}{\lambda} \tag{5}$$

Refer Balanis Antenna theory second edition page no 277 onwards. There is a derivation for end fire directivity also.

From the above formula (5) we have a) $D_0 = 10log_{10}(N/2)$ dB b) $D_0 = 10log_{10}(N/2)$ dB

10. In a linear array of antennas with the same spacing between each element, four isotropic radiating elements are spaced $\frac{\lambda}{4}$ apart. What should the relative phase shift between the elements required for forming the main beam at 60 degrees with the axis of the array?

Solution: Uniform linear Array of N elements radiates in either broad side or end fire directions based on progressive phase shift, α between the excitation sources connected to antenna elements in the Array.The array factor is given by



$$AF = 1 + e^{j\beta d\cos\theta+\alpha} + e^{2(j\beta d\cos\theta+\alpha)} + e^{3(j\beta d\cos\theta+\alpha)} + ...e^{(n-1)(j\beta d\cos\theta+\alpha)}$$

$$AF = \sum_n e^{j(n-1)\psi}$$

$$\psi = \alpha + \beta d\cos\theta$$

Given, the main beam is 60°off end fire i.e. $\theta = 60°$ Substituting the value of θ, we get the value of α which is $\alpha = -\frac{\pi}{4}$

11. 1. **Practice Exercise :** Open your mobile phones ,go to Settings− >About Phone− >Status (or Network) On this screen, view Signal Strength (or Network Type and Strength). Check your signals strength will be in dBm which is a unit of level used to indicate that a power ratio is expressed in decibels (dB) with reference to one milliwatt (mW).Convert the signal strength from dBm to mW.
    2. Suppose a person with a receiver is 5 km away from the transmitter. What is the distance that this person must move further to detect a 3 dB decrease in signal strength?

Solution: From some position P_1 the person having receiver moves some distance to detect 3 dB decrease is signal strength. Power strength at P_2 is $\frac{1}{2}$ times power strength at position 1. As power varies as $\frac{1}{r^2}$.Hence

$$\frac{P_1}{P_2} = \frac{r_2^2}{r_1^2}$$

Solving for r_2,we have distance between initial point and final point to be 2.070 km