EE2025 Engineering Electromagnetics: July-Nov 2019

Tutorial 5: Parallel plane and rectangular waveguides

1. A parallel plane waveguide is filled with a material with dielectric constant 9. The height of the waveguide is 50 cm. At 1 GHz how many modes can be propagated inside the waveguide and what are their cut-off frequencies ?

> **Solution:**
>
> The cut-off frequency in a parallel plate waveguide with separation d between the plates is given by $f_c = \frac{vm}{2d}$, where v is the phase velocity and m is the mode number.
>
> For $f_c = 1GHz$, $m = \frac{2df_c}{v} = \frac{2\times 0.5\times 10^9 \times \sqrt{9}}{3\times 10^8} = 10$. Since any mode with $f > f_c$ will propagate, 19 modes (TE_m and TM_m, m=1,2,...9 and TM_0) can propagate.
> The cut-off frequency for each mode is given, by the equation above, as
>
> $$f_{cm} = \frac{vm}{2d} = m \times 10^8 Hz \quad ; \quad m = 1, 2, ..., 9 \tag{1}$$

2. (a)Derive the expressions for the transverse field components in terms of the longitudinal field components E_z and H_z, in Cartesian and cylindrical coordinate systems.
(b)Using the general formulation, find modes in parallel plate waveguide

> **Solution:**
>
> (a) Refer section 6.3 in R.K.Shevgaonkar's "Electromagnetic waves".
>
> For cylindrical co-ordinate system, the procedure is same. z direction is the longitudinal direction and ρ and ϕ are the transverse directions. Only difference is in the expression of ∇ operator as given below.
>
> Gradient : $\boxed{\bar{\nabla} = \hat{\rho}\frac{\partial}{\partial\rho} + \frac{\hat{\phi}}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}}$
>
> Divergence : $\boxed{\bar{\nabla}\cdot\bar{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(A_\rho\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}}$
>
> Curl : $\boxed{\bar{\nabla}\times\bar{A} = \hat{\rho}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \hat{z}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(A_\phi\rho) - \frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\right)}$
>
> Laplacian : $\boxed{\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}}$
>
> Hence we obtain the following final expressions for fields.
>
> $$E_\rho = \frac{-j\omega\mu}{h^2}\frac{1}{\rho}\frac{\partial H_z}{\partial\phi} + \frac{-j\beta}{h^2}\frac{\partial E_z}{\partial\rho} \tag{2}$$
>
> $$E_\phi = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial\rho} + \frac{-j\beta}{h^2}\frac{1}{\rho}\frac{\partial E_z}{\partial\phi} \tag{3}$$
>
> $$H_\rho = \frac{j\omega\epsilon}{h^2}\frac{1}{\rho}\frac{\partial E_z}{\partial\phi} + \frac{-j\beta}{h^2}\frac{\partial H_z}{\partial\rho} \tag{4}$$
>
> $$H_\phi = \frac{-j\omega\epsilon}{h^2}\frac{\partial E_z}{\partial\rho} + \frac{-j\beta}{h^2}\frac{1}{\rho}\frac{\partial H_z}{\partial\phi} \tag{5}$$

(b) Refer section 6.5 in R.K.Shevgaonkar's "Electromagnetic waves".

3. For a 5 cm × 3 cm rectangular waveguide, the maximum peak electric field of the dominant mode at 5 GHz is 10 V/m. Find the maximum peak magnetic field inside the waveguide. Also find the total power carried by the waveguide.

Solution:

Given, a=5 cm, b=3 cm and f=5 GHz.

Dominant mode $\Rightarrow TE_{10}$ mode $\Rightarrow h = \frac{\pi}{a}$

$\mathbf{E_z} = 0V/m$

$\mathbf{H_z} = C\cos(\frac{\pi}{a}x)e^{-j\beta z}A/m$

$\mathbf{H_x} = -\frac{1}{h^2}(\gamma\frac{\partial \mathbf{H_z}}{\partial x}) = (\frac{j\beta}{h})C\sin(\frac{\pi}{a}x)e^{-j\beta z}A/m$

$\mathbf{H_y} = -\frac{1}{h^2}(\gamma\frac{\partial \mathbf{H_z}}{\partial y}) = 0A/m$

$\mathbf{E_x} = -\frac{1}{h^2}(j\omega\mu\frac{\partial \mathbf{H_z}}{\partial y}) = 0V/m$

$\mathbf{E_y} = -\frac{1}{h^2}(j\omega\mu\frac{\partial \mathbf{H_z}}{\partial x}) = (\frac{j\omega\mu}{h})C\sin(\frac{\pi}{a}x)e^{-j\beta z}V/m$

Given, $|E_y|_{max} = 10v/m \Rightarrow (\frac{\omega\mu}{h})C = 10 \Rightarrow C = \frac{10h}{\omega\mu} = 0.0159$

$\beta = \sqrt{\omega^2\mu\epsilon - h^2} = 83.866$

$|H_x|_{max} = (\frac{\beta}{h})C = 0.0212A/m$

Power density, p= $\frac{1}{2}[\mathbf{E}\times\mathbf{H}^*] = \frac{1}{2}\times 10\times 0.0212\times\sin^2(\frac{\pi x}{a}) = 0.106\sin^2(\frac{\pi x}{a})$

Power carried by waveguide=$\int_0^b\int_0^a pdxdy = 0.106\times b\int_0^a\sin^2(\frac{\pi x}{a})dx = 79.5\mu W$

4. For the fundamental mode inside a rectangular waveguide the logitudinal magnetic field is given as

$$H_z = 20\cos(10y)e^{-j\beta z}A/m$$

Find the cut-off frequency of the mode. Also, find the frequency at which the group velocity is 1/3 of the phase velocity.

Solution:

Given, $H_z = 20\cos(10y)e^{-j\beta z}A/m$

It is in TE_{01} mode. Then, $H_z = C\cos(\frac{\pi}{b}y)e^{-j\beta z}A/m$

$C = 20$, $\frac{\pi}{b} = 10$

cut-off frequency, $f_c = \frac{(\frac{\pi}{b})}{2\pi\sqrt{\mu_0\epsilon_0}} = \frac{10}{2\pi\sqrt{\mu_0\epsilon_0}} = \frac{10}{2\pi}c = 0.478GHz$

Given, group velocity is 1/3 of the phase velocity i.e. $v_g = \frac{1}{3}v_p$

As, $\sqrt{v_p v_g} = c \Rightarrow v_p = \sqrt{3}c \Rightarrow \frac{c}{\sqrt{1-(\frac{f_c}{f})^2}} = \sqrt{3}c$

$f = \sqrt{\frac{3}{2}}f_c = 0.585GHz$

5. A rectangular waveguide has conducting fins running along its length as shown in Figure 1. Find the lowest frequency which will propagate on the waveguides.

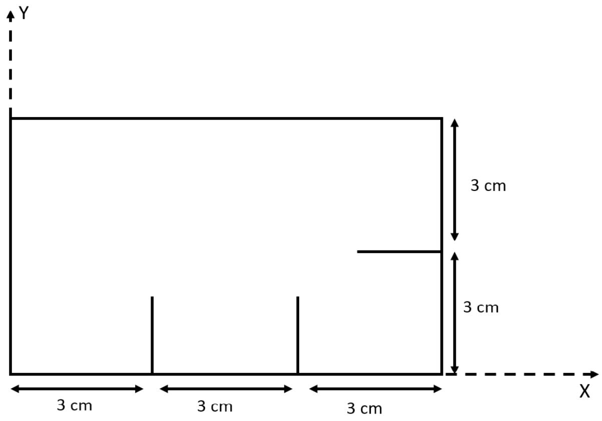

Figure 1

Solution: The conducting fins impose additional boundary conditions on the modal fields. The tangential component of the electric field and normal component of magnetic field must go to zero at the fins.
For any TE mode:
For fins along the x-axis, E_y has to be 0 which means $\sin \frac{m\pi x}{a}$ should go to zero.
As the fins along x-axis are located at $x = 3cm$ and $6cm$ respectively and $a = 9cm$, the value of m has to be an integer multiple of 3 for field component to be 0.

For the fin along the y-axis, E_x has to be 0 which means $\sin \frac{n\pi y}{b}$ should go to zero.
As the fin along y-axis is located at $y = 3cm$ and $b = 6cm$, the value of n has to be an integer multiple of 2 for field component to be 0

Thus, the allowed modes for TE_{mn} are TE_{32},TE_{62},TE_{92},TE_{34},TE_{64},TE_{94} etc.

Calculating for the minimum frequency, TE_{32} has the min. cut-off frequency
$f_c = \frac{c}{2} \times \sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2} = 9.013GHz$

6. Inside an air-filled waveguide, the magnetic field component for a TM mode is given as

$$\mathbf{H} = 10\cos(\pi x)\sin\left(\frac{\pi y}{2}\right)e^{-j\beta z}\hat{y}A/m$$

Find the vector electric field, the phase constant β and the cut off frequency of the wave. The frequency of the wave is 2 GHz.

Solution:

$$\mathbf{H_y} = 10\cos(\pi x)\sin\left(\frac{\pi y}{2}\right)e^{-j\beta z}\hat{y}A/m$$

From the expression, $\frac{m}{a} = 1$ and $\frac{n}{b} = 0.5$
$h^2 = (\pi^2) + (\frac{\pi}{2})^2$
$(\frac{-j\omega\epsilon}{h^2})(\frac{m\pi}{a})C = 10$
$C = j352.95$
$h^2 = \omega^2\mu\epsilon - \beta^2$
$\beta = 41.77rad/m$

Comparing the corresponding electric field equations, we get

$$\mathbf{E_z} = j352.95 \sin(\pi x) \sin\left(\frac{\pi y}{2}\right) e^{-j\beta z} \hat{z} V/m$$

$$\mathbf{E_x} = 3754.11 \cos(\pi x) \sin\left(\frac{\pi y}{2}\right) e^{-j\beta z} \hat{x} V/m$$

$$\mathbf{E_y} = 1877.055 \sin(\pi x) \cos\left(\frac{\pi y}{2}\right) e^{-j\beta z} \hat{y} V/m$$

Cut-off frequency $f_c = 167.705 MHz$

7. Standard air-filled waveguides have been designed for the radar bands. One type, designed WG-16, is suitable for X-band applications. Its dimensions are, $a = 2.29$ cm and $b = 1.02$ cm. If it is desired that a WG=16 waveguide operate only in the dominant TE_{10} mode and that the operating frequency be at least 25 % above the cutoff frequency of the TE_{10} mode but no higher than 95 % of the next higher cutoff frequency, what is the allowable operating frequency range ?

Solution: Cut off frequency for a mode is given by

$$f_c = \frac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

For the air-filled waveguide, the cut off frequency for TE_{10} mode is given by

$$f_c = \frac{c}{2\pi}\left(\frac{\pi}{a}\right) = 6.55 GHz$$

The next higher mode is TE_{20} and the cut off frequency can be calculated to be

$$f_c = \frac{c}{a} = 13.10 GHz$$

The allowable frequency range is $8.18 GHz \leq f \leq 12.4 GHz$

8. A standard air-filled S-band rectangular waveguide has dimensions $a = 7.21$ cm and $b = 3.40$ cm. What mode types can be used to transmit electromagnetic waves having the following wavelengths?
(**a**) $\lambda = 10$ cm
(**b**)$\lambda = 5$ cm

Solution: Cut off wavelength for a mode is given by

$$\lambda_c = \frac{2}{\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2}}$$

All the modes which have cut off wavelengths higher than the wavelength of the excitation will propagate inside the waveguide.
The cut off wavelength for different modes in this waveguide are given below

mode	λ_c (cm)
TE_{10}	14.42
TE_{20}	7.20
TE_{01}	6.8
TE_{11}/TM_{11}	6.1
TE_{02}	3.40

(a) For $\lambda = 10$ cm, the propagating mode is TE_{10}
(b) For $\lambda = 5$ cm, the propagating modes are TE_{10}, TE_{20}, TE_{01}, TE_{11}, TM_{11}

9. Parallel plate waveguide made of two perfectly conducting infinite planes spaced 3 cm apart operates at 10 GHz. Which of the modes (TEM/TE/TM) would be preferable for carrying the largest time average power through this waveguide, without causing dielectric breakdown. Justify by explicitly calculating the largest time average power that can be propagated through these modes.
(Given dielectric breakdown of air occurs at 3×10^6 V/m).

Solution: Poynting vector $\mathcal{P}_{avg} = \frac{1}{2} Re\{\mathbf{E} \times \mathbf{H}^*\}$
Total Power $P_{avg} = \int_{x=0}^{d} \int_{y=0}^{1} |\mathcal{P}_{avg}|\, dxdy$, calculating per unit length in $y-$direction.
For TE mode,

$$E_1 = 2jE_{i0} \sin\left(\frac{m\pi x}{d}\right) e^{-j\beta_1 z}\hat{y} \tag{6}$$

$$H_1 = 2j\frac{m\pi}{\beta_1 d}\frac{E_{i0}}{\eta} \sin\left(\frac{m\pi x}{d}\right) e^{-j\beta_1 z}\hat{x} + 2\frac{\beta}{\beta_1}\frac{E_{i0}}{\eta} \cos\left(\frac{m\pi x}{d}\right) e^{-j\beta_1 z}\hat{z} \tag{7}$$

Then we get,

$$\mathcal{P}_{avg} = 2\frac{|E_{i0}|^2}{\eta}\frac{m\pi}{\beta_1 d} \sin^2\left(\frac{m\pi x}{d}\right)\hat{z} \tag{8}$$

$$\mathcal{P}_{avg} = 2\frac{|E_{i0}|^2}{\eta} \cos\theta \sin^2\left(\frac{m\pi x}{d}\right)\hat{z} \tag{9}$$

$$P_{avg} = \frac{|E_{i0}|^2}{\eta} d \cos\theta \tag{10}$$

where θ is the angle of incidence.(Refer section 6.1 of R. K. Shevgaonkar)
For TM mode,

$$E_1 = 2j\frac{m\pi}{\beta_1 d} E_{i0} \sin\left(\frac{m\pi x}{d}\right) e^{-j\beta_1 z}\hat{z} + 2\frac{\beta}{\beta_1} E_{i0} \cos\left(\frac{m\pi x}{d}\right) e^{-j\beta_1 z}\hat{x} \tag{11}$$

$$H_1 = 2\frac{E_{i0}}{\eta} \cos\left(\frac{m\pi x}{d}\right) e^{-j\beta_1 z}\hat{y} \tag{12}$$

Then we get,

$$\mathcal{P}_{avg} = 2\frac{|E_{i0}|^2}{\eta}\frac{\beta}{\beta_1} \cos^2\left(\frac{m\pi x}{d}\right)\hat{z} \tag{13}$$

$$\mathcal{P}_{avg} = 2\frac{|E_{i0}|^2}{\eta} \sin\theta \cos^2\left(\frac{m\pi x}{d}\right)\hat{z} \tag{14}$$

$$P_{avg} = \frac{|E_{i0}|^2}{\eta} d \sin\theta \tag{15}$$

For TEM mode,

$$E_1 = 2E_{i0}e^{-j\beta_1 z}\hat{x} \tag{16}$$

$$H_1 = 2\frac{E_{i0}}{\eta}e^{-j\beta_1 z}\hat{y} \tag{17}$$

$$\mathcal{P}_{avg} = 2\frac{|E_{i0}|^2}{\eta}\hat{z} \tag{18}$$

$$P_{avg} = 2\frac{|E_{i0}|^2}{\eta}d \tag{19}$$

Observing the P_{avg} of these modes, we can say that the largest time averaged power is carried by TEM mode ($\sin\theta, \cos\theta < 1$).

For the maximum power that can be allowed in a particular mode without breakdown, consider $E_{i0} = 3 \times 10^6$ V/m (dielectric break down of air), $\eta = 377\Omega$, $d = 3$ cm and say $\theta = 60^o$.

$P^{TE}_{avg} = 3.58 \times 10^8$ W/m, $P^{TM}_{avg} = 6.20 \times 10^8$ W/m, $P^{TEM}_{avg} = 14.32 \times 10^8$ W/m

10. An air-filled $a \times b$ ($b < a < 2b$) rectangular waveguide to be constructed to operate at 3 GHz in the dominant mode. We desire the operating frequency to be at least 20 % higher than the cut off frequency of the dominant mode and also atleast 20 % below the cut off frequency of the next higher order mode.
(**a**) Give a typical design for the dimensions a and b
(**b**)Calculate for your design β, u_p, λ_g at the operating frequency
(**c**)Define intrinsic impedance for TE mode. Calculate intrinsic impedance at the operating frequency

Solution: (**a**)We want the waveguide to be operational at 3 GHz in the dominant mode i.e. TE_{10}

Dominant frequency is given by, $f_c = \frac{c}{2}\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2}$

Cutoff frequency of dominant mode,

$$f_c = \frac{c}{2a}$$

Cutoff frequency of the next higher order mode TE_{01} (Since, $a < 2b$ can't be TE_{20}),

$$f_c^{'} = \frac{c}{2b}$$

Given,

$$f_o \geq 1.2 \times f_c$$

$$f_o \leq 0.8 \times f_c^{'}$$

$$a \geq (0.6) \times \frac{c}{3} \times 10^{-9} \geq 6cm$$

$$b \leq (0.4) \times \frac{c}{3} \times 10^{-9} \leq 4cm$$

Thus taking into consideration $a < 2b$ we get,

$$6cm \leq a \leq 2b$$

$$3cm \leq b \leq 4cm$$

Proposed design, $a = 6cm, b = 4cm$

(**b**)

$$\lambda_o = \frac{3 \times 10^8}{3 \times 10^9} = 0.1m$$

$$\lambda_c = 2a = 0.12m$$

$$\lambda_g = \frac{\lambda_o}{\sqrt{1 - (\frac{\lambda_o}{\lambda_c})^2}} = \frac{0.1}{\sqrt{1 - (\frac{0.1}{0.12})^2}} = 0.1809m$$

$$u_p = \frac{c}{\sqrt{1 - (\frac{\lambda_o}{\lambda_c})^2}} = \frac{3 \times 10^8}{\sqrt{1 - (\frac{0.1}{0.12})^2}} = 5.427 \times 10^8 ms^{-1}$$

$$\beta = \frac{2\pi f}{u_p} = \frac{2\pi \times 3 \times 10^9}{5.427 \times 10^8} = 34.732radm^{-1}$$

(**c**) In TE mode after solving the Maxwell's equation subject to boundary conditions the intrinsic impedance works out to be,

$$\eta_{TE} = \frac{\eta_o}{\sqrt{1 - (\frac{\lambda_o}{\lambda_c})^2}}$$

$$\eta_{TE} = \frac{120\pi}{\sqrt{1 - (\frac{\lambda_o}{\lambda_c})^2}} = \frac{120\pi}{\sqrt{1 - (\frac{0.1}{0.12})^2}} = 682\Omega$$

11. A resonant cavity of 6 cm × 3 cm × 4 cm is excited in the lowest mode. The peak electric field inside the cavity is 100 V/m. Find the resonant frequency of the cavity and the total energy stored inside the cavity.

Solution: The lowest mode is TE_{101}; for which the resonant frequency is given by

$$f_0 = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{d}\right)^2} = 4.507GHz \tag{20}$$

The total energy stored as electric field is given by

$$W_e = \frac{\epsilon}{2}E_o^2 \int_0^6 \int_0^3 \int_0^4 sin^2(\frac{\pi x}{a})sin^2(\frac{\pi z}{d})dxdydz = 1.937 \times 10^{-16} Joules \tag{21}$$

This also equals the energy stored as magnetic field. So, the total energy stored equals $2W_e = 3.87 \times 10^{-16} J$