EE2025 Engineering Electromagnetics: July-Nov 2019

Tutorial 3: Plane Waves in Lossy Dielectrics and Media Interface

1. A rectangular copper block is 30 cm in height (along z). In response to a wave incident on the block from above, a current is induced in the block in the positive x direction. Find the ratio of ac resistance of the block to its dc resistance at 1 kHz. Consider $\sigma = 5.8 \times 10^7$ S/m, $\mu_r = 1$.

Solution:

Surface resistance $R_s = \sqrt{\frac{\pi f \mu}{\sigma}} = 8.25\mu\Omega$

$$R_{\text{dc}} = \frac{l}{\sigma S} = \frac{l}{\sigma w h} \text{ and } R_{\text{ac}} = \frac{R_s l}{w}$$

$$\Rightarrow \frac{R_{\text{ac}}}{R_{\text{dc}}} = R_s \sigma h = 143.55$$

2. At 2 GHz, the conductivity of the meat is of the order of 1 S/m. When meat is placed in a microwave oven, the EM field in the conducting material causes energy dissipation in the material in the form of heat. Derive the expression for average power density in the meat, if the peak electric field inside is E_0. Evaluate the average power density (in W/mm^2), for $E_0 = 4 \times 10^4$ V/m.

Solution: Surface resistance $R_s = \sqrt{\frac{\pi f \mu}{\sigma}} = \frac{3.14 \times 2 \times 10^9 \times 4\pi \times 10^{-7}}{1} = 88.85\,\Omega$

Refer to section 4.9 of Shevgaonkar for more the surface current and power density derivations.

Surface current $|J_s| = \frac{\sigma E_0}{\gamma}$

$|\gamma| = \sqrt{\omega \mu_0 \sigma} = 125.66$ m^{-1} $\Rightarrow |J_s| = 318.32$ A/m$= 0.318$ A/mm

Power dissipated $P = \frac{1}{2} R_s |J_s|^2 \approx 4.5$W/mm^2.

3. At microwave frequencies,the power density considered for human exposure is $1mW/cm^2$. A RADAR radiates a wave described as

$$E(r) = \frac{3000}{r} V/m$$

where r is the radial co-ordinate, which is the distance from the source. What is the radius of unsafe region?

Solution: Average power density, $P_{ave} = \frac{1}{2} Re(\mathbf{E} \times \mathbf{H}^*) = \frac{E_0^2}{2|\eta|} e^{-2\alpha z} cos(\theta_\eta)$

For air, $\eta = 377\ \Omega$ and $\theta_\eta = 0$, $\alpha = 0$ for lossless media. Hence we can write, $P_{ave} = \frac{1}{2\eta} E_0^2$

Given, $1mW/cm^2 = \frac{1}{2\eta} E_0^2 = \dfrac{E_0^2}{2 \times 377} \Rightarrow E_0 = 0.868\ V/cm = 86.8\ V/m$

$E(r) = \frac{3000}{r}\ V/m = 86.8 \Rightarrow r = 34.5494\ m$

$r \leq 34.55$ m is the radius of unsafe region.

4. For copper, $\mu = 4\pi \times 10^{-7} H/m$, $\epsilon = \epsilon_0 = \frac{1}{36\pi} \times 10^{-9} F/m$, $\sigma = 5.8 \times 10^7 S/m$. Assuming these are frequency independent, find the range of electromagnetic spectrum for which copper is a good conductor. Is it a good conductor for visible, UV, IR ranges ?

Solution: For a good conductor, $\sigma >> \omega\epsilon$ so that $\frac{\sigma}{\omega\epsilon} >> 1$
Hence $\omega << \frac{\sigma}{\epsilon}$.
i.e. $\omega = 2\pi f << \frac{5.8*10^7*36\pi}{10^{-9}}$
$f << \frac{5.8*10^7*36\pi}{2\pi*10^{-9}} \Rightarrow f << 1.04$ x 10^{18} Hz
This is the X-ray range in the electromagnetic spectrum. The IR, visible and UV frequencies are in the order of 10^{13}, 10^{15} and 10^{16} respectively. Hence copper is a conductor in these frequency ranges.

5. A laptop manufacturer wants to shield her laptop such that the energy from its high frequency clock doesn't radiate to the outside world. Assuming a clock rate of 2.45 GHz, what should be the thickness of the metal cladding on the laptop cover and weight of metal required, given standard laptop dimension of 15"x11" if she uses

   (a) silver ($\sigma = 6.2 * 10^7 S/m$,density=$10.49 g/cm^3$)
   (b) gold ($\sigma = 4.1 * 10^7 S/m$,density=$19.32 g/cm^3$)
   (c) copper ($\sigma = 5.8 * 10^7 S/m$,density=$8.96 g/cm^3$)

Solution: Given $f = 2.45 GHz$
To shield the laptop, thickness of the metal cladding should be at least equal to five times the skin depth where the radiating field strength reduces considerably.
Skin depth for any metal with conductivity σ at frequency f, $\delta = \frac{1}{\sqrt{f\pi\sigma\mu}}$

(a) silver ($\sigma = 6.2 * 10^7 S/m$)
$\delta = \frac{1}{\sqrt{\pi*2.45*10^9*6.2*10^7*4\pi*10^{-7}}} = 1.2913\ \mu m$
Thickness of silver $= 5 * 1.2913 = 6.4565 \mu m$

(b) gold ($\sigma = 4.1 * 10^7 S/m$)
$\delta = \frac{1}{\sqrt{\pi*2.45*10^9*4.1*10^7*4\pi*10^{-7}}} = 1.588\ \mu m$
Thickness of gold $= 5 * 1.588 = 7.94 \mu m$

(c) copper ($\sigma = 5.8 * 10^7 S/m$)
$\delta = \frac{1}{\sqrt{\pi*2.45*10^9*5.8*10^7*4\pi*10^{-7}}} = 1.335\ \mu m$
Thickness of copper $= 5 * 1.335 = 6.675 \mu m$

Reference for density values: https://www.coolmagnetman.com/magconda.htm

Conclusion: choose the cheapest for base models, and lightest for deluxe models!

6. In a dielectric medium, a wave has electric and magnetic fields given as,

$$E = (j\hat{x} + 2\hat{y} - j\hat{z})\exp[-j\pi(x+z)]V/m$$

$$H = \frac{1}{60\pi}(-\hat{x} + j\hat{y} + \hat{z})\exp[-j\pi(x+z)]A/m$$

   Show that the wave is a uniform plane wave. Find

   (a) phase constant of the wave
   (b) velocity of the wave
   (c) frequency of the wave

Solution: A uniform plane wave follows the below relations;
$\mathbf{k}.\mathbf{E} = 0$
$\mathbf{k}.\mathbf{H} = 0$
$\mathbf{a_E} \times \mathbf{a_H} = \mathbf{a_k}$
From the field expressions, we get
$\mathbf{k} = \pi\hat{x} + \pi\hat{z}$
Solving for the conditions,
$\mathbf{k}.\mathbf{E} = (\pi\hat{x} + \pi\hat{z}).(j\hat{x} + 2\hat{y} - j\hat{z}) \exp\left[-j\pi(x+z)\right] = (j\pi - j\pi) \exp\left[-j\pi(x+z)\right] = 0$
$\mathbf{k}.\mathbf{H} = (\pi\hat{x} + \pi\hat{z}).\frac{1}{60\pi}(-\hat{x} + j\hat{y} + \hat{z}) \exp\left[-j\pi(x+z)\right] = (-\pi + \pi) \exp\left[-j\pi(x+z)\right] = 0$
As **E** and **H** have constant phase difference between them, we can say, it's a uniform plane wave.

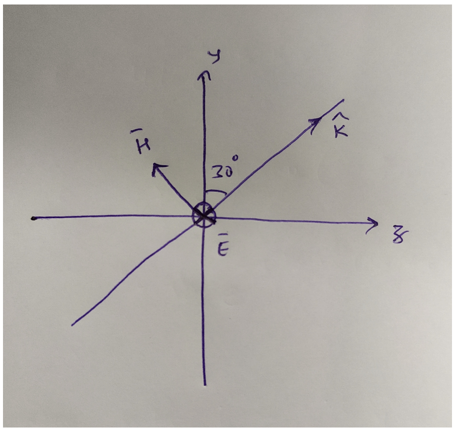
(a) phase constant of the wave
$\beta = \sqrt{(k_x)^2 + (k_z)^2} = \sqrt{2}\pi rad/s$

(b) velocity of the wave
$|\eta| = \frac{|\mathbf{E}|}{|\mathbf{H}|} = \sqrt{2} \times 60\pi\Omega = \sqrt{\frac{\mu}{\epsilon}} = \frac{1}{\sqrt{\epsilon_r}} \times 120\pi$
$\sqrt{\epsilon_r} = \sqrt{2}$
$v = \frac{c}{\sqrt{\epsilon_r}} = 2.12 \times 10^8 m/s$

(c) frequency of the wave
$\frac{\omega}{\beta} = v$
$f = \frac{\beta \times v}{2\pi} = 150MHz$

7. A plane wave travels in yz-plane at an angle of 30° from the +y direction. The electric field of the wave is oriented in the x-direction with an amplitude 50 V/m. If the medium has the dielectric constant of 2.5. Find the vector magnetic field, the wave vector and the phase constant of the wave. Frequency of the wave is 1GHz.

Solution:



Given, $\mu_r = 1$ $\epsilon_r = 2.5$ f=1GHz,
Phase constant, $\beta = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c}\sqrt{\epsilon_r} = 33.12\, m^{-1}$
wave vector, $\hat{k} = \cos(30°)\mathbf{a}_y + \sin(30°)\mathbf{a}_z = \frac{\sqrt{3}}{2}\mathbf{a}_y + \frac{1}{2}\mathbf{a}_z$
$\mathbf{k} = \beta\hat{k} = 33.12(\frac{\sqrt{3}}{2}\mathbf{a}_y + \frac{1}{2}\mathbf{a}_z) = 16.56(\sqrt{3}\mathbf{a}_y + \mathbf{a}_z)$
Electric field, $\mathbf{E} = E_0 e^{j(\omega t - \mathbf{k}.\mathbf{r})}\mathbf{a_E} = 50e^{j[(2\pi \times 10^9 t) - 16.56(\sqrt{3}y + z)]}\mathbf{a_x}$ V/m

$\eta = \sqrt{\frac{\mu}{\epsilon}} = 120\pi\sqrt{\frac{1}{2.5}} = 238.43\,\Omega$

$H_0 = \frac{E_0}{\eta} = 0.209$ A/m

Magnetic field, $\mathbf{H} = H_0 e^{j[(2\pi\times10^9 t)-16.56(\sqrt{3}y+z)]}[\sin(30°)\mathbf{a_y} - \cos(30°)\mathbf{a_z}]$

$\mathbf{H} = 0.105 e^{j[(2\pi\times10^9 t)-16.56(\sqrt{3}y+z)]}(\mathbf{a_y} - \sqrt{3}\mathbf{a_z})$ A/m

8. A non magnetic medium has an intrinsic impedance of $240\angle 30\ \Omega$. Find its
   (a) loss tangent
   (b) Dielectric constant
   (c) Complex permittivity
   (d) Attenuation constant at 1 MHz.

Solution:

$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} = |\eta|\angle\theta_\eta = 240\angle 30$

Where,

$|\eta| = \frac{\sqrt{\mu/\epsilon}}{[1+(\frac{\sigma}{\omega\epsilon})^2]^{1/4}} = 240$

$\theta_\eta = \frac{1}{2}\tan^{-1}(\frac{\sigma}{\omega\epsilon}) = 30°$

a) loss tangent$=(\frac{\sigma}{\omega\epsilon}) = \tan(2\theta_\eta) = \sqrt{3} = 1.732$

b) From, $|\eta| = \frac{120\pi\sqrt{\mu_r/\epsilon_r}}{[1+(\frac{\sigma}{\omega\epsilon})^2]^{1/4}} = 240$

$\sqrt{\frac{1}{\epsilon_r}} = \frac{240}{120\pi}[1+(\sqrt{3})^2]^{1/4}$

dielectric constant=1.234

c) complex permittivity$=(1.091 - j1.89)\times 10^{-11}$ F/m

d)attenuation constant at f=1MHz is

$\alpha = \frac{\omega}{c}\sqrt{\frac{\mu_r\epsilon_r}{2}[\sqrt{1+(\frac{\sigma}{\omega\epsilon})^2} - 1]} = 0.0164$ Np/m

9. The electric field component of an EM wave propagating in air is

$$\mathbf{E} = 50\cos(\omega t - \beta_1 x \sin 45° - \beta_1 z \cos 45°)\ \mathbf{a_y} V/m$$

If the wave is incident on a lossless medium ($\epsilon = 2.25\epsilon_0, \mu = \mu_0$) in $z \geq 0$, determine the magnetic field component, the transmission coefficient and Brewster angle.

Solution: XZ plane is the plane of incidence and **E** field (in Y direction) is perpendicular to it. Hence this is the case of wave with perpendicular polarization at a media interface.

$$\mathbf{H} = \frac{1}{\omega\mu}\mathbf{k}\times\mathbf{E} = \frac{\hat{k}\times\mathbf{E}}{\eta}$$

$$\mathbf{k} = \beta_1\hat{k} = \beta_1(\sin 45°\hat{\mathbf{a_x}} + \cos 45°\hat{\mathbf{a_z}})$$

$$\mathbf{H} = \frac{50}{\eta}(-\cos 45^\circ \hat{\mathbf{a_x}} + \sin 45^\circ \hat{\mathbf{a_z}})\cos(\omega t - \beta_1 x \sin 45^\circ - \beta_1 z \cos 45^\circ) A/m$$

$$\mathbf{H} = 93.78(-\hat{\mathbf{a_x}} + \hat{\mathbf{a_z}})\cos(\omega t - \beta_1 x \sin 45^\circ - \beta_1 z \cos 45^\circ) mA/m$$

From Snell's law,

$$\sqrt{\mu_1 \epsilon_1}\sin\theta_i = \sqrt{\mu_2 \epsilon_2}\sin\theta_t$$

We have $\theta_i = 45^\circ$, $\mu_1 = \mu_2 = \mu_0$, $\epsilon_1 = \epsilon_0$ and $\epsilon_2 = 2.25\epsilon_0$

$$\sin\theta_t = \sqrt{\frac{\epsilon_1}{\epsilon_2}}\sin\theta_i = \sqrt{\frac{1}{2.25}}\sin 45^\circ$$

$$\theta_t = 28.12^\circ$$

Transmission coefficient for perpendicular polarisation

$$\tau_\perp = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

$$\eta_1 = 120\pi, \eta_2 = \frac{120\pi}{\sqrt{2.25}}$$

$$\tau_\perp = 0.696$$

Brewster angle does not exist for perpendicular polarisation in the case of non magnetic materials (Refer to section 5.7 of R K Shevgaonkar)

10. A plane wave from free space to a lossless dielectric with $\mu = \mu_0, \epsilon = 4\epsilon_0$ is totally transmitted. Find θ_i and θ_t. What is the state of polarization of the wave ?

Solution: There is no reflection when the angle of incidence is Brewster angle. For non magnetic materials ($\mu_1 = \mu_2 = \mu_0$), Brewster angle exist only for parallel polarisation.

$$\theta_i = \theta_{B||} = \tan^{-1}\left(\sqrt{\frac{\epsilon_2}{\epsilon_1}}\right) = 63.43^\circ$$

From Snell's law,

$$\sqrt{\mu_1 \epsilon_1}\sin\theta_i = \sqrt{\mu_2 \epsilon_2}\sin\theta_t$$

We have $\theta_i = 63.43^\circ$, $\mu_1 = \mu_2 = \mu_0$, $\epsilon_1 = \epsilon_0$ and $\epsilon_2 = 4\epsilon_0$

$$\sin\theta_t = \sqrt{\frac{1}{4}}\sin 63.43^\circ$$

$$\theta_t = 26.56^\circ$$

The state of polarisation is parallel polarisation (parallel with respect to the plane of incidence)

11. A parallel polarized wave in air with, $\mathbf{E} = (6\mathbf{a_y} - 8\mathbf{a_z})\sin(\omega t - 4y - 3z)$ V/m impinges a dielectric half space as shown in Figure 1. Find

(a) Incidence angle θ_i.

(b) Time average power in air ($\epsilon_0 = 8.85 * 10^{-12} F/m, \mu_0 = 4\pi * 10^{-7} H/m$).

(c) The Reflected and Transmitted Electric fields (E_r, E_t).

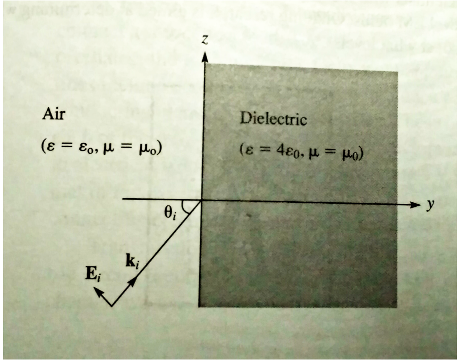


Figure 1: Air and Dielectric half spaces

Solution:

(a)

$$\mathbf{k_i} = 4\hat{\mathbf{a_y}} + 3\hat{\mathbf{a_z}}$$

θ_i=angle between $(\mathbf{k_i}, \mathbf{a_n})$, $(\hat{\mathbf{a_y}})$ is normal to boundary surface $(\hat{\mathbf{a_z}})$

$$|\mathbf{k_i}| \cos\theta_i = 4$$

$|\mathbf{k_i}| = 5$, Incidence angle

$$\theta_i = 36.86^\circ$$

(b)

Time averaged power =

$$\frac{1}{2} Re(E \times H^*) = \frac{|E_{i0}|^2}{2\eta} \hat{\mathbf{a_k}} = \left(\frac{8^2 + 6^2}{2 \times 120\pi}\right) \frac{4\hat{\mathbf{a_y}} + 3\hat{\mathbf{a_z}}}{5} = 0.106\hat{\mathbf{a_y}} + 0.079\hat{\mathbf{a_z}}$$

(c)

Equating tangential component of propagation Vectors with parameters or using Snell's law at boundary $\mu_1 = \mu_2 = \mu_0$, $\epsilon_1 = \epsilon_0$ and $\epsilon_2 = 4$.

Angle of Transmission

$$\sqrt{\mu_1 \epsilon_1} \sin\theta_i = \sqrt{\mu_2 \epsilon_2} \sin\theta_t$$

$$\sin\theta_t = 0.3$$

$$\theta_t = 17.45^\circ$$

Angle of Reflection

$$\sqrt{\mu_1 \epsilon_1} \sin\theta_i = \sqrt{\mu_1 \epsilon_1} \sin\theta_r$$

$$\sin\theta_r = 0.6$$

$$\theta_r = 36.86^\circ$$

Reflection coefficient and Reflected Electric field

$$\Gamma_{||} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_1\cos\theta_i + \eta_2\cos\theta_t}$$

$$\eta_1 = 120\pi\,\Omega,\ \eta_2 = 60\pi\,\Omega$$

$$\Gamma_{||} = \frac{0.9539 - 1.6}{1.6 + 0.9539}$$

$$\Gamma_{||} = -0.25$$

$$\mathbf{k_r} = -|\mathbf{k_r}|\cos\theta_r\hat{\mathbf{a_y}} + |\mathbf{k_r}|\sin\theta_r\hat{\mathbf{a_z}}$$

$$|\mathbf{k_r}| = |\mathbf{k_i}| = 5$$

$$\mathbf{k_r} = -4\hat{\mathbf{a_y}} + 3\hat{\mathbf{a_z}}$$

$$\mathbf{E_r} = -0.25(-6\hat{\mathbf{a_y}} - 8\hat{\mathbf{a_z}})\sin(\omega t + 4y - 3z)V/m$$

Transmission coefficient And Transmitted Electric field

$$\tau_{||} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$$\tau_{||} = 0.63$$

transmitted Propagation constant

$$\mathbf{k_t} = |\mathbf{k_t}|\cos\theta_t\hat{\mathbf{a_y}} + |\mathbf{k_t}|\sin\theta_r\hat{\mathbf{a_z}}$$

$$|\mathbf{k_t}|\sin\theta_t = |\mathbf{k_i}|\sin\theta_i = 5 * \frac{3}{5}$$

$$\frac{3}{10}|\mathbf{k_t}| = 5 * \frac{3}{5}$$

$$|\mathbf{k_t}| = 10$$

$$\mathbf{k_t} = 9.54\hat{\mathbf{a_y}} + 2.99\hat{\mathbf{a_z}}$$

$$\mathbf{E_t} = 0.63(6\hat{\mathbf{a_y}} - 8\hat{\mathbf{a_z}})\sin(\omega t - 9.54y - 2.98z)V/m.$$

12. A 1 GHz electromagnetic wave is normally incident on a 3 cm thick plastic slab of dielectric constant 5. What percent of the incident power is:

    (a) transmitted out through the slab ?

    (b) reflected back out off the slab ?

Solution: This is a case of multiple reflections and transmissions. (Refer 5.4.1 in Shevgaonkar)
Region 1 assumed to be air

$$\eta_1 = \eta_o$$

Region 2 is plastic(non-magnetic) of dielectric constant 5,

$$\eta_2 = \sqrt{\frac{\mu_o}{\epsilon}} = \frac{\eta_o}{\sqrt{5}}$$

Reflection and transmission coefficients are written out as,

$$\Gamma_{12} = \frac{(\frac{\eta_o}{\sqrt{5}} - \eta_o)}{(\frac{\eta_o}{\sqrt{5}} + \eta_o)} = \frac{1-\sqrt{5}}{1+\sqrt{5}}$$

$$\Gamma_{23} = -\Gamma_{12} = \frac{\sqrt{5}-1}{\sqrt{5}+1}$$

$$\tau_{12} = \frac{2(\frac{\eta_o}{\sqrt{5}})}{\frac{\eta_o}{\sqrt{5}} + \eta_o} = \frac{2}{\sqrt{5}+1}$$

$$\tau_{23} = \frac{2\eta_o}{\frac{\eta_o}{\sqrt{5}} + \eta_o} = \frac{2\sqrt{5}}{\sqrt{5}+1}$$

Propagation constant in the plastic slab,

$$\beta_2 = \frac{\omega_o}{c}\sqrt{\mu_r \epsilon_r} = \frac{2\pi \times 10^9}{3 \times 10^8}\sqrt{5} = \frac{20\sqrt{5}\pi}{3}$$

$$\beta_2 d = \frac{20\sqrt{5}\pi}{3}(0.03) = 0.2\sqrt{5}\pi$$

Transmission coefficient,

$$\tau = \frac{(\frac{2}{\sqrt{5}+1})(\frac{2\sqrt{5}}{\sqrt{5}+1})(e^{-j(0.2\sqrt{5}\pi)})}{1-(\frac{\sqrt{5}-1}{\sqrt{5}+1})(\frac{\sqrt{5}-1}{\sqrt{5}+1})(e^{-j(2(0.2\sqrt{5}\pi))})} = \frac{0.8541(e^{-j(0.2\sqrt{5}\pi)})}{1-0.145898(e^{-j(2(0.2\sqrt{5}\pi))})}$$

$$\tau = \frac{0.14099 - 0.98628j}{1.1379 + 0.0475j}$$

Reflection Coefficient,

$$\Gamma = \frac{1-\sqrt{5}}{1+\sqrt{5}} + \frac{(\frac{2}{\sqrt{5}+1})(\frac{\sqrt{5}-1}{\sqrt{5}+1})(\frac{2\sqrt{5}}{\sqrt{5}+1})(e^{-j(0.2\sqrt{5}\pi)})}{1-(\frac{1-\sqrt{5}}{1+\sqrt{5}})(\frac{\sqrt{5}-1}{\sqrt{5}+1})(e^{-j(2(0.2\sqrt{5}\pi))})}$$

$$\Gamma = -0.3819 + \frac{0.3262(e^{-j(0.2\sqrt{5}\pi)})}{1+0.145898(e^{-j(2(0.2\sqrt{5}\pi))})}$$

$$\Gamma = -0.3819 + \frac{0.05384 - 0.32172j}{0.862 - 0.0475j} = \frac{-0.2753 - 0.30357j}{0.862 - 0.0475j}$$

The percentage of power transmitted out through the slab is $|\tau|^2 = (0.8748)^2 = 76.527\%$
The percentage of power reflected back by the slab is $|\Gamma|^2 = (0.474698)^2 = 22.53\%$