EE2025 Engineering Electromagnetics: July-Nov 2019

Tutorial 2: Maxwell's equations & Plane waves

1. Prove that electric and magnetic fields obey the principle of super position. *Hint*: Use Maxwell's equations to show this.

> **Solution:** Say that sources $\rho_i, \vec{J_i}$ produce fields $\vec{E_i}$ and $\vec{H_i}$ for $i = 1, 2$. Then we can substitute into Maxwell's equations to show that the fields produced due to a linear combination of the sources is a linear combination of the respective fields.
>
> [uday]

2. In the class, we said that the 'lab quantity' $e(t)$ can be written in terms of it's phasor E as, $e(t) = \text{Re}(Ee^{j\omega t})$. However, the choice of $e^{j\omega t}$ over $e^{-j\omega t}$ is just a convention, i.e. we could have also chosen $e(t) = \text{Re}(Ee^{-j\omega t})$. Using this information, find out whether the following waves are forward traveling or backward traveling waves.

   (a) $e^{-j\omega t}$ convention, $E(z) = Ae^{-jkz}$

   (b) $e^{j\omega t}$ convention, $E(z) = Ae^{-jkz}$

   (c) $e^{j\omega t}$ convention, $E(z) = Ae^{-jkz} + Be^{jkz}$

   where $A, B \in \mathbb{C}$ are constants.

> **Solution:** To determine whether a wave is forward or backward traveling we need to look at the locus of constant phase and find the phase velocity v_p.
>
> For example, consider (a) $e^{-j\omega t}$ convention, $E(z) = Ae^{-jkz}$.
>
> $$\begin{aligned} e(z,t) &= \text{Re}(E(z)e^{-j\omega t}) \\ &= \text{Re}(Ae^{-j(\omega t + kz)}) \\ &= |A| \cos(\omega t + kz - \phi_A) \end{aligned}$$
>
> where, ϕ_A is the phase of A. Let $\psi(z,t)$ be the phase of e(z,t). Locus of constant phase implies $\frac{\partial \psi(z,t)}{\partial t} = 0$
>
> $$\begin{aligned} \psi(z,t) &= \omega t + kz - \phi_A \\ \frac{\partial \psi(z,t)}{\partial t} &= \omega + kv_p = 0 \\ \implies v_p &= -\omega/k \end{aligned}$$
>
> Since $v_p < 0$, this is a **backward traveling wave**.
>
> Alternatively, consider the phase $\psi(z,t) = \omega t - kz - \phi_A$. At $t = 0, z = 0$, $\psi(0,0) = -\phi_A$.
>
> At $t = t_0, \quad t_0 > 0$, for the phase to still be $-\phi_A$ (i.e. locus of constant phase), $z = -\omega t_0/k$. This proves that it is a backward traveling wave. Similarly, for the other questions:
>
> (b) We get $v_p > 0$. Therefore it is a forward traveling wave.
>
> (c) $E(z) = Ae^{-jkz} + Be^{jkz}$ is not a traveling wave. It is a standing wave composed of a forward and a backward traveling wave.
>
> [Karteek]

3. Using the differential form of Maxwell's equations, derive the continuity equation ($\nabla . \vec{J} = -\frac{\partial \rho}{\partial t}$).

Solution: Consider, $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$. Taking divergence on both sides of the equation,

$$\nabla.(\nabla \times \vec{H}) = \frac{\partial \nabla.\vec{D}}{\partial t} + \nabla.\vec{J}$$

We know that $\nabla.\vec{D} = \rho$. Further, divergence of a curl is zero ($\nabla.(\nabla \times \vec{H}) = 0$). Substituting this in the above equation, we get the continuity equation $\nabla.\vec{J} = -\frac{\partial \rho}{\partial t}$.

[Karteek]

4. Given a plane wave travelling in the $+x$ direction in free space has the following phasor expressions for the electric and magnetic fields: $\vec{E} = \hat{y}E_y(x) + \hat{z}E_z(x)$ and $\vec{H} = \hat{y}H_y(x) + \hat{z}H_z(x)$. Draw a vector diagram showing all these components, and indicate which of the E's are related to which of the H's by a constant of proportionality.

Solution: We are dealing with a forward travelling wave in the $+x$ direction, so using the right hand thumb rule relation between $\vec{E}$ and $\vec{H}$ gives us that:

$$E_y/H_z = \eta = -E_z/H_y$$

This is important, because for e.g. there will be no relation between E_y and H_y. One can think of this as the super position of two waves.

[uday]

5. If the electric and magnetic field in a medium are given by $\vec{E} = 3\sin(t - 5z)\,\hat{x}$ and $\vec{H} = 4\cos(t - 5z)\,\hat{y}$, then calculate (at $z = 0$) the
a) the instantaneous power density,
b) instantaneous power transmitted through a surface with an area of $5\,m^2$ at $z = 0$ and the normal pointing in $\hat{z}$ direction, and
c) total energy carried by the wave through the given surface from $t = 0\,s$ to $t = 5\,s$.

Solution: a) The **instantaneous** power density at $z = 0$ is given by

$$\begin{aligned}\vec{S} &= \vec{E} \times \vec{H} \\ &= 3\sin(t) \times 4\cos(t)\,\hat{z} = 12\sin(t)\cos(t)\,\hat{z}\end{aligned}$$

Note: In this question, you have been asked to compute the instantaneous power density. Therefore, you should not use the phasor formula.

b) Power through an area of $5\,m^2$ is given by

$$P = \int \vec{S} \cdot \hat{n}\, ds = 60\sin(t)\cos(t)$$

c) Total energy carried by the wave is given by

$$E = \int_0^5 P(t)\,dt = \int_0^5 60\sin(t)\cos(t)\,dt = 15(1 - \cos(10))$$

[Siddhant]

6. In a non-magnetic material ($\mu_r = 1$) with dielectric constant $\epsilon_r = 4$, the electric field is given by $\vec{e}(t) = 20\sin(10^8 t - \beta z)\hat{y}$. Calculate the propagation constant β and the magnetic field $\vec{h}(t)$.

Solution: We know that

$$\begin{aligned}\beta &= \omega\sqrt{\mu\epsilon} \\ &= 10^8\sqrt{\mu_0 \times 4\epsilon_0} = \frac{10^8(2)}{3\times 10^8} = \frac{2}{3}\end{aligned}$$

Now converting the electric field into phasor form : $\vec{e}(t) = Im(\vec{E}e^{j\omega t}) \Rightarrow \vec{E} = 20e^{-j\beta z}\hat{y}$.
From the Maxwell's equation

$$\begin{aligned}\nabla\times\vec{E} &= -j\omega\mu\vec{H} \\ \vec{H} &= \frac{\nabla\times\vec{E}}{-j\omega\mu} \\ &= \frac{1}{-j\omega\mu}\left[-\frac{\partial E_y}{\partial z}\hat{x}\right] \\ &= -\frac{20\beta}{\omega\mu}e^{-j\beta z}\hat{x}\end{aligned}$$

Substituting the value of β, we get

$$\vec{H} = -\frac{1}{3\pi}e^{-j\beta z}\hat{x}$$

So the sinusoidal form of magnetic field is given by

$$\begin{aligned}\vec{h}(t) &= Im(\vec{H}_s e^{j\omega t}) \\ &= -\frac{1}{3\pi}\sin(10^8 t - \beta z)\hat{x}\end{aligned}$$

[Siddhant]

7. The magnetic field component of a plane wave in a lossless dielectric is $\vec{H} = 30\sin(2\pi\times 10^8 t - 5x)\,\hat{z}$ mA/m.

(a) if $\mu_r = 1$, find ϵ_r

(b) Calculate the wave length and wave velocity.

(c) Determine the intrinsic impedance

(d) Determine the polarization of the wave.

(e) Find the corresponding electric field component.

(f) calculate the Poynting vector

Solution:

(a)

$$\begin{aligned}\beta &= \frac{\omega}{c}\sqrt{\epsilon_r} \\ \sqrt{\epsilon_r} &= \frac{\beta c}{\omega} = \frac{5\times 3\times 10^8}{2\pi\times 10^8} = \frac{15}{2\times\pi} \\ \Longrightarrow\ \epsilon_r &= 5.6993\end{aligned}$$

(b)

$$\lambda = \frac{2\times\pi}{\beta} = \frac{2\times\pi}{5} = 1.2566m.$$
$$u = \frac{c}{\sqrt{\mu_r\epsilon_r}} = \frac{3\times10^8}{\frac{15}{2\pi}} = 1.257\times10^8 m/s.$$

(c)

$$\eta = \eta_0\sqrt{\frac{\mu_r}{\epsilon_r}} = \frac{120\pi}{\frac{15}{2\pi}} = 157.91\Omega$$

(d)

$$\begin{aligned} \hat{a_E}\times\hat{a_H} &= \hat{a_k} \\ \implies \hat{a_E}\times\hat{a_z} &= \hat{a_x} \\ \implies \hat{a_E} &= \hat{a_y} \end{aligned}$$

(e)

$$\begin{aligned} \vec{E} &= \eta H\hat{a_E} \\ &= 30\times10^{-3}(157.91)\sin{(2\pi\times10^8 t - 5x)}\hat{a_E} \\ &= 4.737\sin{(2\pi\times10^8 t - 5x)}\hat{a_y} V/m \end{aligned}$$

(f) The Poynting vector,

$$\vec{P} = \vec{E}X\vec{H}$$
$$\implies \vec{P} = 0.1421\ \sin^2{(2\pi\times10^8 t - 5x)}\ \hat{a_x}\ W/m^2$$

[Naresh]

8. Consider a x-directed time varying electric field propagating in z direction $E_x(x,y,z) = f(x,y) + g(z)$ in source-free, free space. Using Maxwell's equations, prove that if it is a plane wave, then $f(x,y) = 0$.

Solution:

For a plane wave,time-varying fields are constant in a plane containing the field vector.Hence the field should be constant in x-y plane or x-z pane,which implies it should have no x-dependance.Hence $f(x,y) = 0$

Consider the equation

$$\nabla\times\vec{E} = -j\omega\mu\vec{H}$$
$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = -j\omega\mu\vec{H}$$

$$\implies \frac{\partial E_x}{\partial z}\hat{y} + \frac{\partial E_x}{\partial y}\hat{z} = -j\omega\mu(\vec{H}_y\hat{y} + \vec{H}_z\hat{z}) \tag{1}$$

Considering the other Maxwell's equation with $\vec{J} = 0$, we have

$$\nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t}$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & H_y & H_z \end{vmatrix} = \epsilon \frac{\partial \vec{E}}{\partial t}$$

$$\implies (\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z})\hat{x} - \frac{\partial H_z}{\partial x}\hat{y} + \frac{\partial H_y}{\partial x}\hat{z} = \epsilon \frac{\partial E_x}{\partial t}\hat{x} \tag{2}$$

Hence from equations (1) and (2),we can see that for the plane wave to exist, there should be no component along z direction.Hence $f(x, y) = 0$

[Aggraj]

9. What values of A and β are required if the two fields given below satisfy Maxwell's equations in a linear, isotropic, homogeneous medium with $\epsilon_r = \mu_r = 4$ and $\sigma = 0$?

$$\vec{E} = 120\pi cos(10^6\pi t - \beta x)\hat{a_y} V/m$$
$$\vec{H} = A\pi cos(10^6\pi t - \beta x)\hat{a_z} V/m$$

Assume there are no current or charge densities in space.

Solution:

$$\vec{E} = 120\pi cos(10^6\pi t - \beta x)\hat{a_y} V/m$$
$$\vec{H} = A\pi cos(10^6\pi t - \beta x)\hat{a_z} V/m$$

Method 1

$$\frac{|E|}{|H|} = 120\pi$$
$$\longrightarrow A = \frac{1}{\pi} = 0.318$$

From the expressions, $\omega = 10^6\pi$

$$\frac{\omega}{\beta} = (3 \times 10^8)/4$$
$$\longrightarrow \beta = 0.0418 rad$$

Method 2
From Maxwell's equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t}$$
$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} = \epsilon \frac{\partial \vec{E}}{\partial t}$$
$$\nabla \times \vec{E} = 120\pi\beta sin(\omega t - \beta x)\hat{a_z} \tag{3}$$

$$-\mu\frac{\partial \vec{H}}{\partial t} = \mu A 10^6 \pi^2 sin(\omega t - \beta x)\hat{A}_z \tag{4}$$

From (3) and (4),

$$120\beta = A\mu 10^6 \pi$$

$$\frac{\beta}{A} = \frac{\mu 10^6 \pi}{120} \tag{5}$$

$$\nabla \times \vec{H} = -A\pi\beta sin(\omega t - \beta x)\hat{a_y} \tag{6}$$

$$\epsilon\frac{\partial \vec{E}}{\partial t} = -120\epsilon\pi^2 10^6 sin(\omega t - \beta x)\hat{a_y} \tag{7}$$

From (6) and (7),

$$A\beta = 120\epsilon\pi 10^6 \tag{8}$$

Using (5) and (8)
A=0.318
$\beta = 0.041866$ rad
[$Prajosh$]

10. Consider a point on the surface of a perfect conductor. The electric field intensity at that point is $\vec{E} = (500\hat{x} - 300\hat{y} + 600\hat{z})\cos 10^7 t$ and medium surrounding the conductor is characterized by $\mu_r = 5$ and $\epsilon_r = 10$ and $\sigma = 0$.
(a) Find a unit vector normal to the conductor at that point of the conductor surface.
(b) Find the surface charge density at the point.

Solution: (a) Since we have a perfect conductor,Electric field at any point will be normal and directed out of the surface.So it will bw given by outward normal electric field vector at that point.Hence,

$$\hat{n} = \frac{\vec{E}(t=0)}{|E(t=0)|}$$

$$\implies \hat{n} = \frac{5\hat{x} - 3\hat{y} + 6\hat{z}}{\sqrt{(5^2 + 9^2 + 36)}} = 0.60\hat{x} - 0.36\hat{y} + 0.72\hat{z} \tag{9}$$

(b) Since it is a metallic conductor,no field will exist inside the metal.Hence with boundary conditions we have, Surface charge density is given by

$$(\vec{D_1}.\hat{n} - \vec{D_2}.\hat{n}) = \rho_s$$

So here D_2 will be zero as it is zero within the metallic conductor so,the net surface charge density will be just

$$(\vec{D_1}.\hat{n}) = \rho_s$$

$$\implies \rho_s = 10\epsilon_0(500\hat{x} - 300\hat{y} + 600\hat{z}).(0.60\hat{x} - 0.36\hat{y} + 0.72\hat{z}) = 74\cos 10^7 t \tag{10}$$

[Aggraj]

11. Assume two regions are separated by $z = 0$ plane. Let $\mu_1 = 4$ μH/m in region 1 where $z > 0$, while $\mu_2 = 7$ μH/m in region 2 wherever $z < 0$. We are given $\mathbf{B}_1 = 2\mathbf{a}_x - 3\mathbf{a}_y$ mT, in region 1. Find $\mathbf{B}_2$ for both the cases (a) and (b). a) Let surface current be $\mathbf{J}_s = 80\mathbf{a}_x$ A/m on $z = 0$. b) Let surface current be $\mathbf{J}_s = 80\mathbf{a}_z$ A/m on $z = 0$.

Solution: a) The normal component of $\mathbf{B}_1$ is 0
We determine the tangential components:

$\mathbf{B}_{t1} = 2\mathbf{a}_x - 3\mathbf{a}_y$ mT

and

$\mathbf{H}_{t1} = \frac{\mathbf{B}_{t1}}{\mu_1} = 500\mathbf{a}_x - 750\mathbf{a}_y$ A/m

Thus, $\mathbf{a}_n$ is the unit normal vector, and $\mathbf{a}_n$ is -$\mathbf{a}_z$ for the surface $z = 0$.

$\mathbf{H}_{t2} = \mathbf{H}_{t1} - \mathbf{a}_n \times \mathbf{J}_s = 500\mathbf{a}_x - 750\mathbf{a}_y - (-\mathbf{a}_z) \times 80\mathbf{a}_x$ A/m

$\mathbf{H}_{t2} = 500\mathbf{a}_x - 670\mathbf{a}_y$ A/m

and

$\mathbf{B}_{t2} = \mu_2\mathbf{H}_{t2} = 3.5\mathbf{a}_x - 4.69\mathbf{a}_y$ mT

Therefore,

$\mathbf{B}_2 = \mathbf{B}_{t2} = 3.5\mathbf{a}_x - 4.69\mathbf{a}_y + \mathbf{a}_z$ mT

b) For the surface current $\mathbf{J}_s = 80\mathbf{a}_z$

$\mathbf{a}_n \times \mathbf{J}_s = 0$. Hence,

$\mathbf{H}_{t2} = \mathbf{H}_{t1}$

[chandan]

12. Consider the result of superimposing left and right circularly polarized fields of the same amplitude, frequency and propagation direction, but where a phase shift of δ radians exists between two. What is the polarization of the resultant field? (consider the wave is travelling in +z direction)

Solution:

$\mathbf{E}_R = \mathbf{E}_{rc} + \mathbf{E}_{lc} = E_0[\mathbf{a}_x - j\mathbf{a}_y]e^{-j\beta z} + E_0[\mathbf{a}_x + j\mathbf{a}_y]e^{-j\beta z}e^{j\delta}$

Grouping components together,

$\mathbf{E}_R = E_0[(1 + e^{j\delta})\mathbf{a}_x - j(1 - e^{j\delta})\mathbf{a}_y]e^{-j\beta z}$

Factoring out an overall phase term, $e^{j\delta/2}$

$\mathbf{E}_R = E_0e^{j\delta/2}[(e^{-j\delta/2} + e^{j\delta/2})\mathbf{a}_x - j(e^{-j\delta/2} - e^{j\delta/2})\mathbf{a}_y]e^{-j\beta z}$

This can be simplified as

$\mathbf{E}_R = 2E_0[\cos(\delta/2)\mathbf{a}_x + \sin(\delta/2)\mathbf{a}_y]e^{-j(\beta z - \delta/2)}$ Linearly Polarized wave, whose field vector is oriented at an angle $\delta/2$ from x axis

[chandan]

13. We have seen in class that a good conductor is classified by the condition that $\epsilon''/\epsilon' \gg 1$, where the permittivity $\epsilon = \epsilon' - j\epsilon''$. For such a conductor, find a simple relationship for α and β. Further, derive the expression for phase velocity as a function of conductivity σ and frequency f. How does velocity vary with frequency, and how does it vary with conductivity?

Solution: $\alpha = \beta = \sqrt{\omega\mu_0\sigma/2}$ and phase velocity $v_p = c\sqrt{4\pi f\epsilon_0/\sigma}$. Velocity increases with frequency and decreases with conductivity. Also see problem 4.11 of RKS book.

[uday]

14. A uniform plane wave is travelling in seawater. Assume that the x-y plane resides just below the sea surface and the wave travels in the +z direction into the water. The constitutive parameters of seawater are $\epsilon_r = 80$, and $\sigma = 4\ S/m$. If the magnetic field at $z = 0$ is $\boldsymbol{H}(0,t) = \hat{\boldsymbol{y}}\,100\,cos(2\pi * 10^3 t + 15°)$ (m A/m),

(**a**) obtain expressions for $\boldsymbol{E}(z,t)$ and $\boldsymbol{H}(z,t)$, and

(**b**) determine the depth at which the magnitude of $\boldsymbol{E}$ is 1% of its value at $z = 0$.

Solution: **(a)** Since $\boldsymbol{H}$ is along $\hat{\boldsymbol{y}}$ and the propagation direction is $\hat{\boldsymbol{x}}$. Hence, the general expressions for the phasor fields are

$$\widetilde{\boldsymbol{E}}(z) = \hat{\boldsymbol{x}} E_{x0} e^{-\alpha z} e^{-j\beta z},$$
$$\widetilde{\boldsymbol{H}}(z) = \hat{\boldsymbol{y}} \frac{E_{x0}}{\eta_c} e^{-\alpha z} e^{-j\beta z} \ .$$

To determine α, β, and η_c for seawater, we begin by evaluating the ratio $\frac{\epsilon''}{\epsilon'}$. From the argument of the cosine function of $\hat{\boldsymbol{H}}(0,t)$, we deduce that $\omega = 2\pi * 10^3\ (rad/s)$, and therefore $f = 1\ kHz$. Hence,

$$\frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon} = \frac{\sigma}{\omega\epsilon_r\epsilon_0} = \frac{4}{2\pi * 10^3 * 80 * (10^{-9}/36\pi)} = 9 * 10^5\ .$$

This qualifies seawater as a good conductor at 1kHz and allows us to use the following expression for the conductor:

$$\alpha = \sqrt{\pi f \mu\sigma} = \sqrt{\pi * 10^3 * 4\pi * 10^{-7} * 4}$$
$$= 0.126\ (rad/m),$$
$$\beta = \alpha = 0.126\ (rad/m),$$
$$\eta_c = (1+j)\frac{\alpha}{\sigma} = (\sqrt{2}e^{j\pi/4}\frac{0.126}{4}) = 0.044e^{j\pi/4}\ (\Omega).$$

As no explicit information has been given about the electric field amplitude E_{x0}, we should assume it to be complex; that is, $E_{x0} = |E_{x0}|e^{j\phi_0}$. The wave's instantaneous electric and magnetic fields are given by

$$\boldsymbol{E}(z,t) = \Re\left[\hat{\boldsymbol{x}}\,|E_{x0}|e^{j\phi_0}e^{-\alpha z}e^{-j\beta z}e^{j\omega t}\right]$$
$$= \hat{\boldsymbol{x}}\,|E_{x0}|e^{-0.126z}cos(2\pi * 10^3 t - 0.126z + \phi_0)\ (V/m),$$
$$\boldsymbol{H}(z,t) = \Re\left[\hat{\boldsymbol{y}}\,\frac{|E_{x0}|e^{j\phi_0}}{0.044e^{j\pi/4}}e^{-\alpha z}e^{-j\beta z}e^{j\omega t}\right]$$
$$= \hat{\boldsymbol{y}}\,22.5|E_{x0}|e^{-0.126z}cos(2\pi * 10^3 t - 0.126z + \phi_0 - 45°)\ (A/m).$$

At $z = 0$,

$$\boldsymbol{H}(0,t) = \hat{\boldsymbol{y}}\,22.5|E_{x0}|cos(2\pi * 10^3 t + \phi_0 + 45°)\ (A/m).$$

By comparing this expression with the one given in the problem statement,

$$\boldsymbol{H}(0,t) = \hat{\boldsymbol{y}}\,100\,cos(2\pi * 10^3 + 15°)\ (mA/m),$$

we deduce that,

$$22.5|E_{x0}| = 100 * 10^{-3}$$

or

$$|E_{x0}| = 4.44\ (mV/m),$$

and

$$\phi_0 - 45° = 15° \implies \phi_0 = 60°\ .$$

Hence, the final expressions for $\boldsymbol{E}(z,t)$ and $\boldsymbol{H}(z,t)$ are

$$\boldsymbol{E}(z,t) = \hat{\boldsymbol{x}}\,4.44e^{-0.126z}cos(2\pi * 10^3 t - 0.126z + 60°)\ (mV/m),$$

$$\boldsymbol{H}(z,t) = \hat{\boldsymbol{y}}\,100e^{-0.126z}cos(2\pi * 10^3 t - 0.126z + 15°)\ (mA/m).$$

(b) The depth at which the amplitude of $\boldsymbol{E}$ has decreased to 1% of its initial value at $z = 0$ is obtained from: $0.01 = e^{-0.126z}$ or $z = \frac{\ln(0.01)}{-0.126} = 36.55mm \approx 37$mm.

[Uday]