EE2025 Engineering Electromagnetics: July-Nov 2019

Tutorial 1: Transmission Lines

Note : All transmission lines can be assumed to be lossless, unless mentioned otherwise.

1. Sinusoidally varying voltages and currents can in general be represented as $V\cos(\omega t+\psi)$ and $I\cos(\omega t+\phi)$, where V, I are real. These can also be written in phasor notation as $Re[Ve^{j\psi}e^{j\omega t}]$ and $Re[Ie^{j\phi}e^{j\omega t}]$: we now call the terms accompanying $e^{j\omega t}$ as the *phasors* corresponding to the voltage and current respectively (i.e. $Ve^{j\psi}$ and $Ie^{j\phi}$). Note that phasors are always time independent. Find an expression for the average power (over a cycle) in terms of these phasors.

Solution: Let the voltage be represented as $v(t) = V\cos(\omega t+\psi)$, and current as $i(t) = I\cos(\omega t+\phi)$, where V, I are real numbers. Note that these are very general expressions. The instantaneous power is then $p(t) = v(t)\,i(t) = VI\cos(\omega t+\psi)\cos(\omega t+\phi)$. The average power over a cycle then becomes $P = \int_0^T p(t)dt/T = VI\int_0^T \frac{1}{2}(\cos(2\omega t+\psi+\phi)+\cos(\psi-\phi))\,dt/T$. Since $\cos(2\omega t)$ integrates out to zero over a period, we get $P = \frac{1}{2}VI\cos(\psi-\phi)$. We want to express all of this in terms of phasors now.
It is easy to see that we can also write the above as $v(t) = Re[Ve^{j\psi}e^{j\omega t}]$ and $i(t) = Re[Ie^{j\phi}e^{j\omega t}]$, and so the corresponding phasors are $\tilde{V} = Ve^{j\psi}$ and $\tilde{I} = Ie^{j\phi}$ and the expression for average power becomes in terms of phasors becomes $P = \frac{1}{2}Re[\tilde{V}\tilde{I}^*]$. You can check that no other combination works.

2. The length of a microstrip trace line connecting two components on a chip is 50 cm. A sinusoidal signal of frequency 1 GHz is supplied to the trace at one end. Assuming the velocity of propagation of the signal is 2×10^8 m/sec and there are no reflections,

   1. Calculate the time taken by the signal to reach the other end of the trace.
   2. What is the phase difference between the signal at the two ends of the trace ?

Solution: Given details $l = 50$ cm, $f = 1$ GHz and $v = 2\times10^8$ m/sec

1. Transit time $t_r = l/v = \frac{50\times10^{-2}}{2\times10^8} = 2.5$ns
2. additional phase

$$\phi = \beta l = \frac{\omega l}{v} = \frac{2\pi\times10^9\times50\times10^{-2}}{2\times10^8} = 5\pi\,\text{rad}$$

3. Using the concepts of electrostatics, find the capacitance per unit length, C of

   1. parallel wire line, with each wire of radius a and separated by a distance $2d$, where $a << 2d$.
   2. coaxial cable of inner radius a and outer radius b.

Solution:

1. Parallel wire line :

   A two wire transmission line is shown in figure. 1a

   Suppose $d > a$. A charge q^+ is induced on conductor A and q^- is induced on conductor B (per unit length) by a voltage source. The potential in the region between the wires is the sum of the potential due to individual wires.

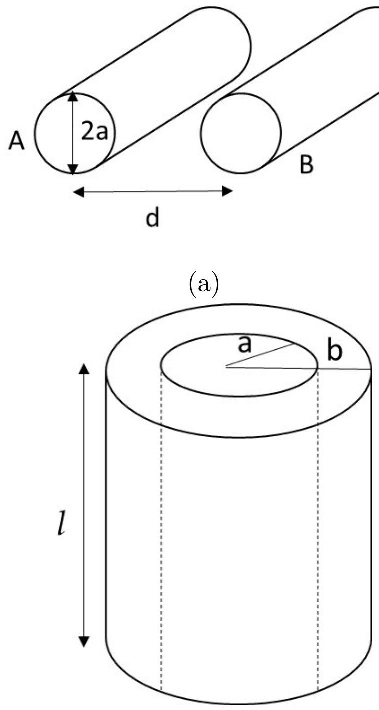


Figure 1: (a)Two wire transmission line. $a << d$. (b) Coaxial transmission line with inner radius a and outer radius b and length l

$$V_A = \int_a^d \frac{q}{2\pi\epsilon r}dr = \frac{q}{2\pi\epsilon}ln(\frac{d}{a})$$

$$V_B = \int_d^a \frac{-q}{2\pi\epsilon r}dr = \frac{-q}{2\pi\epsilon}ln(\frac{a}{d})$$

$$V = V_A + V_B = \frac{q}{2\pi\epsilon}ln(\frac{d}{a}) + \frac{-q}{2\pi\epsilon}ln(\frac{a}{d}) = \frac{q}{\pi\epsilon}ln(\frac{d}{a})$$

Capacitance per unit length, $C = q/V$

$$C = \frac{\pi\epsilon}{ln(\frac{d}{a})}$$

2. Coaxial cable :

   Electric field in the dielectric between the inner and outer conductor in a coaxial line as shown in Figure. 1b is given by

$$\mathbf{E} = \frac{\rho_l}{2\pi\epsilon r}\hat{a_r}$$

   where ρ_L is the charge per unit length and r is the perpendicular distance from the conductor. If the axis of of the cylinder is z axis, potential difference between the conductors,

$$V = -\int_b^a \frac{\rho_l}{2\pi\epsilon r}dr = \frac{\rho_l}{2\pi\epsilon}ln(\frac{b}{a}) = \frac{Q}{l}\frac{1}{2\pi\epsilon}ln(\frac{b}{a})$$

Capacitance, C is,

$$C = \frac{Q}{V} = \frac{2\pi\epsilon l}{ln(\frac{b}{a})}$$

Capacitance per unit length is,

$$C = \frac{2\pi\epsilon}{ln(\frac{b}{a})}$$

4. You are required to buy a cable from an electronics shop to connect your dish antenna to your set top box and your set top box to your TV.

   1. Write the name of the cable you would buy.
   2. Upto what length do you think you can use this cable, in the lumped circuit model and why ?

Solution:

1. co-axial cable
2. We choose the length of the cable such that the product of transit time t_r and frequency of operation f is < 0.1. Table. 1, shows the frequency bands and their corresponding frequency range of the signals, received by the dish antenna.

   Considering the largest frequency $f = 806$ MHz, for the calculations

   $$t_r f < 0.1 \Rightarrow \frac{lf}{v} < 0.1 \Rightarrow l < 1.24 \times 10^{-10} v$$

   For $v = 2 \times 10^8$ m/sec, the length of the cable required is $l = 24.8$ mm. For any length > 24.8 mm, at the chosen frequency of operation the cable behaves like a transmission line.

Frequency Band	**RF Channels**	**Frequency (MHz)**
Very high frequency-Low	2 – 6	54 – 88
Very high frequency-High	7 – 13	174 – 216
Ultra high frequency	14 – 69	470 – 806

Table 1: Frequency bands in television

5. A transmission line with characteristic impedance Z_0 = 50 - j5 Ω and propagation constant γ = 0.2 + j2.5 /m is connected to a load impedance of 100 + j50 Ω. Find

   1. Reflection coefficient of the line at the load end.
   2. Reflection coefficient of the line $5m$ from the load.

Solution:

1. Reflection coefficient of the line at the load end,

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(100 + j50) - (50 - j5)}{(100 + j50) + (50 - j5)} = \frac{50 + j55}{100 + j45} = (0.4067 + j0.2446)$$

.

2. Reflection coefficient of the line 5m from the load,

$$\Gamma(l) = \Gamma_L e^{-2\gamma l} = (0.4067 + j0.2446)\ e^{-2(0.2+j2.5)(5)} = (0.4067 + j0.2446)\ e^{-2} e^{-j25}$$
$$= (0.0501 + j0.0399)$$

.

6. (a) Show that the impedance along the line will lie between Z_0/ρ and $Z_0\rho$, where ρ is the VSWR.
(b)A 300 Ω transmission line is connected to a circuit with an input impedance of 75+j35 Ω. Find

   1. ρ
   2. Maximum impedance seen on the line
   3. Minimum impedance seen on the line

Solution: **(a)** The impedance at any point on the loss-less transmission line is given by,

$$Z(l) = \frac{V(l)}{I(l)} = Z_0 \left[\frac{1 + \Gamma_L e^{-j2\beta l}}{1 - \Gamma_L e^{-j2\beta l}}\right]$$

The extreme values are obtained by exponential giving values 1 and -1.

$$[Z(l)]_{max} = \frac{V_{max}}{I_{min}} = Z_0 \left[\frac{1 + |\Gamma_L|}{1 - |\Gamma_L|}\right] = Z_0\rho$$

$$[Z(l)]_{min} = \frac{V_{min}}{I_{max}} = Z_0 \left[\frac{1 - |\Gamma_L|}{1 + |\Gamma_L|}\right] = Z_0/\rho$$

(b) Given, $Z_0 = 300\Omega$, $Z_L = 75 + j35\Omega$

Reflection co-efficient,

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{75 + j35 - 300}{75 + j35 + 300} = 0.6046\angle 14.1739$$

1. VSWR of the line

$$\rho = \frac{V_{max}}{V_{min}} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = \frac{1 + 0.6046}{1 - 0.6046} = 4.058$$

2. Maximum impedance seen on the line =

$$\rho Z_0 = 4.058 * 300\Omega = 1.2174k\Omega$$

3. Minimum impedance seen on the line =

$$\frac{Z_0}{\rho} = \frac{300}{4.058} = 73.928\Omega$$

7. An RG-59U coaxial cable has a loss of 10 dB per 100 ft of length. A 10 V - 3 A signal is generated using a function generator and connected to one end of the 50 ft long cable. On the other side, the cable is impedance matched to a set top box unit. Find the power delivered to the load.

Solution: Given,

Voltage at source end, V_s = 10 V.

Current at source end, I_s =3 A.

The load end of the coaxial cable is impedance matched.

Loss coefficient of the coaxial cable = 10 dB per 100 ft.

Total loss in 50 ft long coaxial cable = $10(\frac{50}{100}) = 5$ dB.

Then,

Voltage at load end:

$-20log(\frac{V_l}{V_s}) = 5 \Rightarrow V_l = 10 \times 10^{-(\frac{5}{20})} = 5.62$ V.

Current at load end:

$-20log(\frac{I_l}{I_s}) = 5 \Rightarrow I_l = 3 \times 10^{-(\frac{5}{20})} = 1.69$ A.

Power delivered to the load $=\frac{1}{2}\ 5.62 \times 1.69 = 4.75$ W.

8. According to the maximum power transfer theorem, maximum time averaged power is transferred from a source with internal impedance Z_g to a load, Z_L when $Z_g = Z_L^*$. A 50 MHz generator with an internal impedance (Z_g) of 50 Ω is connected to an impedance 50−j25 Ω. How would you ensure maximum power transfer in this case using a lossless transmission line of characteristic impedance 100 Ω, and what should be the minimum length of the transmission line element ? Assume $v = 2 \times 10^8$m/s as wave velocity.

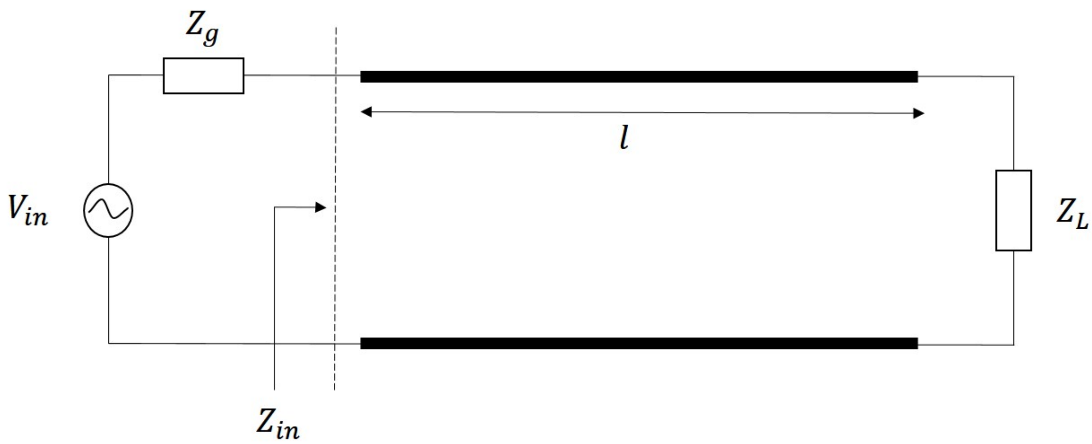


Figure 2: Impedance matching using a transmission line of length l

Solution: Impedance matching can be achieved by inserting a transmission line of length l such that the impedance seen at the input end, Z_{in} is equal to Z_g as shown in Figure. 2

i.e.

$$Z_{in} = Z_o \frac{Z_L + jZ_o tan(\beta l)}{Z_o + jZ_L tan(\beta l)} = 50\,\Omega$$

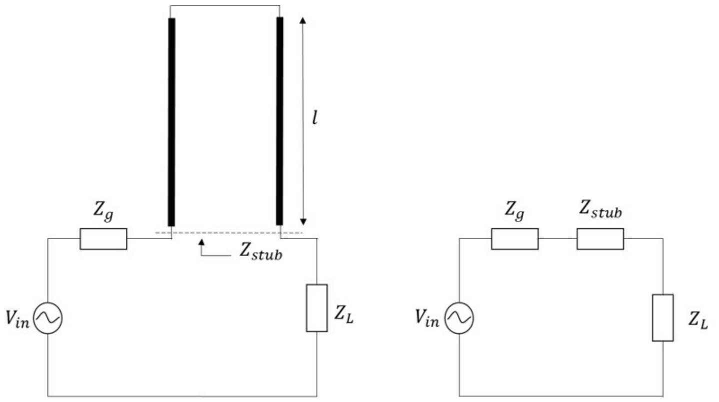


Figure 3: Impedance matching using a short circuited stub of length l and its equivalent circuit

Given, $Z_o = 100\ \Omega$, $Z_L = 50 - j25\ \Omega$

$$\frac{Z_{in}}{Z_o} = \frac{Z_L + jZ_o tan(\beta l)}{Z_o + jZ_L tan(\beta l)} = \frac{50}{100} = \frac{1}{2}$$

$$2Z_L + 2jZ_o tan(\beta l) = Z_o + jZ_L tan(\beta l)$$

$$100 - j50 + 200jtan(\beta l) = 100 + j(50 - j25)tan(\beta l)$$

$$-j50 + j150tan(\beta l) = 25tan(\beta l)$$

Equating real parts, we get

$tan(\beta l) = 0$, Or $\beta l = n\pi \Rightarrow l = \lambda\ /2$; since $\beta = 2\pi/\lambda$

Equating imaginary parts, we get,

$tan(\beta l) = 1/3 \Rightarrow l = 0.051\lambda$

Here, both the solutions should be compatible. i.e., the length of the transmission line should be such that both conditions are satisfied. Since it is not satisfying, we can not implement impedance matching in this method. Another method is to connect a short circuited stub in between the source and load as shown in Figure. 3. Here,

$$Z_{stub} = Z_{sc} = jZ_o tan(\beta l)$$

This Z_{stub} is in series with Z_L. Now the input impedance as seen by the source is the sum of the stub and load impedance. i.e.

$Z_{in} = Z_{stub} + Z_L$

For impedance matching, $Z_{in} = Z_{stub} + Z_L = 50\ \Omega$

$jZ_o tan(\beta l) + 50 - j25 = 50$

$jZ_o tan(\beta l) = j25 \Rightarrow 100tan(\beta l) = 25$

$\beta l = \tan^{-1}(1/4) \Rightarrow \beta l = 0.245$

$l = 0.04\lambda$

9. On a 50 Ω BNC cable line, the reflection co-efficient is measured at the load end to be $0.7\angle 30$. If the propagation constant of the line is $20\angle 89\ /m$, find the impedance seen on the transmission line at a distance of 4 m from the load. (Note : BNC is a very popular type of coaxial cable used

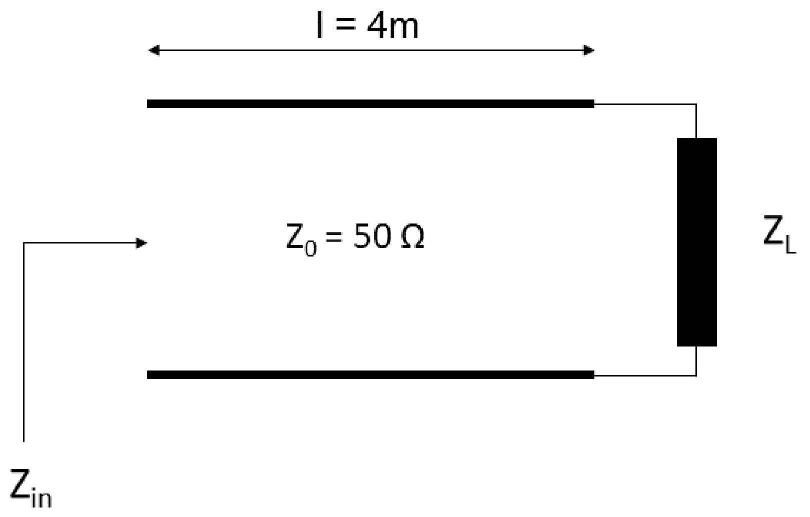
for frequencies even up to 4 GHz)

Solution:

The reflection co-efficient of the line at the load end Γ_l = 0.7∠30 = 0.6062 +j0.35
Z_0 = 50 Ω
As we know,

$$\Gamma_l = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$Z_L = Z_0\left[\frac{1+\Gamma_l}{1-\Gamma_l}\right] = 50\left[\frac{1+0.6062+j0.35}{1-0.6062-j0.35}\right] \simeq 156\angle 54\Omega = 91.6945 + j126.2066\Omega$$



Above figure shows the equivalent circuit diagram for input impedance at a distance of 4m from the load end.
Input impedance at any given distance l from the load is given by the following expression,

$$Z_{in} = Z_o\left[\frac{Z_L + Z_o tanh(\gamma l)}{Z_o + Z_L tanh(\gamma l)}\right]$$

γ= 80∠89 /m
l = 4m
γl = 1.3962+j79.9878
From the expansion of

$$tanh(x+ijy) = \frac{sinh(2x)+sin(2y)}{cosh(2x)+cos(2y)}$$

we get, tanh(1.3962+j79.9878) = 1.1211+j0.0472

$$Z_{in} = 50\left[\frac{91.6945+j126.2066+56.055+j2.36}{50+96.8424+j145.8121}\right] = (47.2216 - j3.1166)\Omega$$

10. Calculate the average power dissipated by each resistor in the circuit shown in Fig. 4.

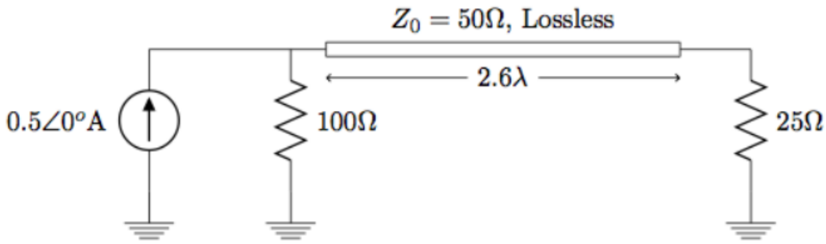
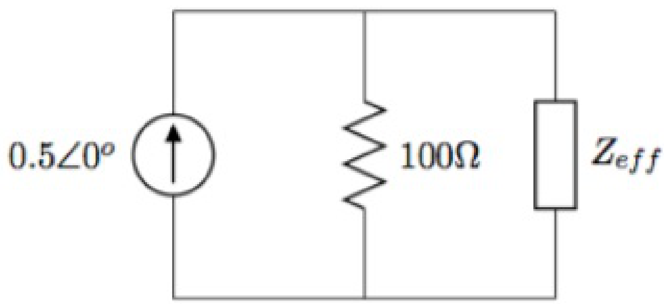


Figure 4

Solution: The circuit is equivalent



Z_{eff}=input impedance of the transmission line terminated in $25\Omega = 50(\frac{25+j50tan(360\times 2.6)}{50+j25tan(360\times 2.6)})$
= 22.3186-j7.3812Ω
The current through the 100Ω resistor $= I_1 = 0.5(\frac{22.3186-j7.3812}{100+22.3186-j7.3812}) =$ 0.0927-j0.02458A
Thus average power dissipated in $100\Omega = P_{100} = \frac{1}{2}|I_1|^2$ x 100 =0.399 W
The voltage across the current source = voltage across $100\Omega = I_1$ x 100 = 9.27-j2.458
Average Power dissipated in source = $P_s = \frac{1}{2}$ $Re[V_s I_s^*]$ = 9.27x0.5 =4.635 W
Thus power dissipated in Z_{eff} = P_s - P_{100} = 4.236 W

11. Given the system in (Fig. 5) is operating with $\lambda = 100cm$ and $Z_0 = 300\Omega$. If $d_1 = 10cm$, $d = 25cm$, and the system is matched to 300 Ω, find Z_L?

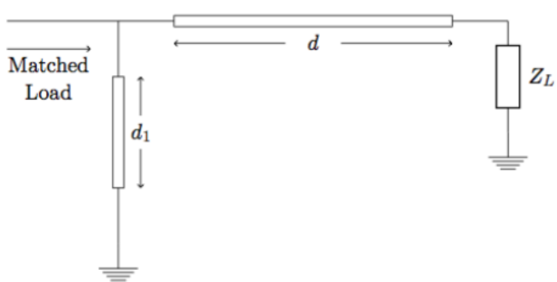
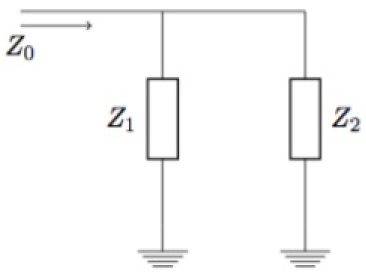


Figure 5

Solution: The equivalent circuit is



Z_1 = impedance of the stub = impedance of a 10cm line shorted = jZ_0tan(βd_1) = j300tan($\frac{2\pi}{100}$10) = j217.963 $Z_0 = Z_1 || Z_2 = \frac{Z_1 Z_2}{Z_1+Z_2}$ and hence $Z_2 = \frac{Z_1 Z_0}{Z_1 - Z_0}$ = 103.64-j142.658Ω
But Z_2 = input impedance of line terminated in $Z_L = Z_0(\frac{Z_L + jZ_0 tan(\beta d)}{Z_0 + jZ_L tan(\beta d)})$
Thus $Z_L = Z_0(\frac{Z_2 - jZ_0 tan(\beta d)}{Z_0 - jZ_2 tan(\beta d)})$ = 300 + j412.91Ω

12. The two-wire lines shown in Fig. 6 are all lossless and have $Z_0 = 200\Omega$. Find the possible values of d and d_1 to provide a matched load if $\lambda = 100cm$. (Note that the un-shaded and shaded conductor are both parts of the same transmission line, for example they can be the inner and outer conductor of a coaxial cable.)

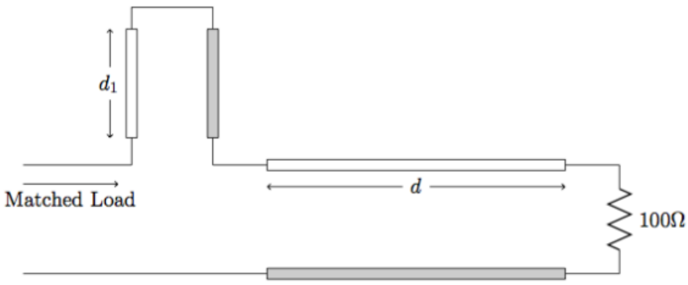
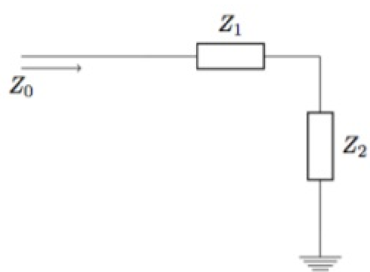


Figure 6

Solution:

The equivalent circuit is as in Fig 12



[h]

Z_1 = impedance of line shorted = jZ_0tan(βd_1)
Z_2 = input impedance of line terminated in 100Ω = $Z_0(\frac{Z_L + jZ_0 tan(\beta d)}{Z_0 + jZ_L tan(\beta d)})$
Thus Z_0 = jZ_0tan(βd_1) + $Z_0(\frac{Z_L + jZ_0 tan(\beta d)}{Z_0 + jZ_L tan(\beta d)})$
Equating real and imaginary parts of both sides:

$1 = \frac{Z_L Z_0(1+tan^2(\beta d))}{Z_0^2+Z_L^2 tan^2(\beta d)}$
$\tan(\beta d_1) = \frac{(Z_L^2-Z_0^2)tan(\beta d)}{Z_0^2+Z_L^2 tan^2(\beta d)}$
$\tan(\beta d) = \pm\sqrt{\frac{Z_0}{Z_L}} = \pm\sqrt{2}$
d = 871.144 cm
$\tan(\beta d_1) = \mp(\sqrt{\frac{Z_0}{Z_L}} - \sqrt{\frac{Z_L}{Z_0}}) = \mp(\sqrt{2} - \sqrt{\frac{1}{2}})$
$d_1 = 561.25$ cm

13. Approximate distributed circuit models of (lossless) a lossless transmission operating in high frequency modes is shown in Fig. 7. Note that L has units $H \cdot m$, C has units $F \cdot m$, L_0 has units H/m and C_0 has units F/m. Obtain expressions for the propagation constant β and the characteristic impedance Z_0 of the line for both circuits at frequency ω.

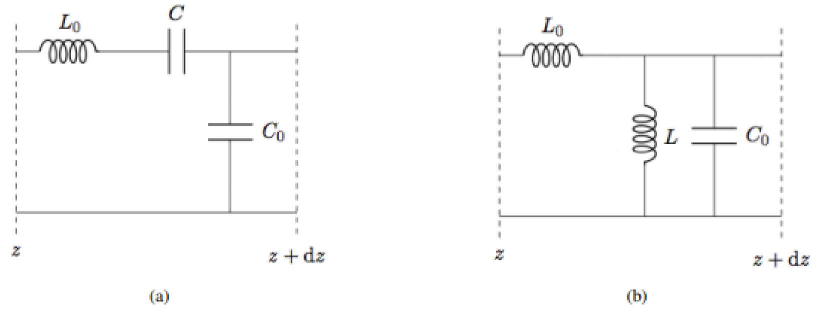


Figure 7

Solution: Assuming initial voltage across capacitors and current in inductors is zero, after taking Fourier transform of time domain equation we get, differential equation of voltage and current for lossless transmission line as

$$\frac{\partial^2 V(z,\omega)}{\partial z^2} = \beta^2 V(z,\omega) and \frac{\partial^2 I(z,\omega)}{\partial z^2} = \beta^2 I(z,\omega)$$

1.

$$V(z,\omega) = j\omega L_0 \Delta z I(z,\omega) + \frac{\Delta z}{j\omega C} + V(z+\Delta z,\omega)$$

$$-\frac{(V(z+\Delta z,\omega) - V(z,\omega))}{\Delta z} = j\omega L_0 I(z,\omega) + \frac{1}{j\omega C}$$

$$-\frac{\partial V(z,\omega)}{\partial z} = j\omega L_0 I(z,\omega) + \frac{1}{j\omega C}$$

$$V(z+\Delta z,\omega) = \frac{1}{j\omega C_0} \frac{I(z,\omega) - I(z+\Delta z,\omega)}{\Delta z}.$$

$$\frac{\partial I(z,\omega)}{\partial z} = -j\omega C_0 V(z,\omega)$$

Second derivative of voltage differential equation gives

$$\frac{\partial^2 V(z,\omega)}{\partial z^2} = \omega^2 L_0 C_0 (1 - \frac{1}{L_0 \omega^2 C}) V(z,\omega) = \beta^2 V(z,\omega).$$

characteristic impedance

$$Z_0(\omega) = \frac{V(z,\omega)}{I(z,\omega)} = \frac{\beta}{\omega C_0} = \sqrt{(\frac{L_0}{C_0(1 - \frac{1}{C\omega^2 L_0})})}$$

2.

$$V(z,\omega) = j\omega L_0 \Delta z I(z,\omega) + V(z+\Delta z,\omega)$$

$$-\frac{(V(z+\Delta z,\omega) - V(z,\omega))}{\Delta z} = j\omega L_0 I(z,\omega)$$

$$-\frac{\partial V(z,\omega)}{\partial z} = j\omega L_0 I(z,\omega)$$

$$V(z+\Delta z,\omega) = \frac{\frac{j\omega L}{\Delta z}\frac{1}{j\omega C_0 \Delta z}}{\frac{j\omega L}{\Delta z} + \frac{1}{j\omega C_0 \Delta z}}(I(z,\omega) - I(z+\Delta z,\omega))$$

$$\frac{\partial I(z,\omega)}{\partial z} = -\frac{1-\omega^2 L C_0}{\omega L} V(z,\omega)$$

Second derivative of voltage differential equation gives

$$\frac{\partial^2 V(z,\omega)}{\partial z^2} = \omega^2 L_0 C_0 (1 - \frac{1}{L\omega^2 C_0}) V(z,\omega) = \beta^2 V(z,\omega).$$

characteristic impedance

$$Z_0(\omega) = \frac{V(z,\omega)}{I(z,\omega)} = \frac{\omega L_0}{\beta} = \sqrt{(\frac{L_0}{C_0(1 - \frac{1}{L\omega^2 C_0})})}$$

14. For the transmission line represented in Fig. 8, calculate the potential developed across the 80Ω resistor for (a) $f = 60Hz$, (b) $f = 1MHz$, (c) Repeat part (a) with length $10^7 m$ instead of $80m$.

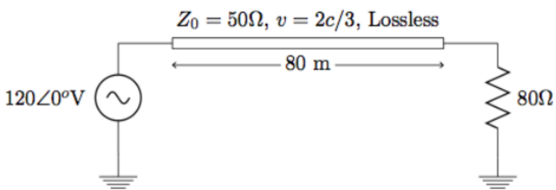


Figure 8

Solution:

$$\lambda = \frac{v}{f} = \frac{2c}{3f}$$
$$\beta = \frac{2\pi}{\lambda} = \frac{3\pi f}{c}$$
$$\Gamma_L = \frac{Z_L - Z_o}{Z_L - Z_o} = \frac{80-50}{80+50} = \frac{3}{13}$$

Voltage at any point a distance z from the source on a lossless transmission line is given by,

$$V(z) = V_+(e^{j\beta z} + \Gamma_L e^{-j\beta z})$$

At z = 0, at the source $V(0) = 120\,V$, substituting the same,

$$V(0) = V_+(1 + \Gamma_L)$$
$$V_+ = \frac{120}{(1+\frac{3}{13})} = 97.5\,V$$

Voltage across the load V_L is given by substituting $z = l$ in the voltage equation,

$$V_L = 97.5(e^{j\beta l} + \frac{3}{13}e^{-j\beta l})$$
$$V_L = 97.5(e^{j\frac{3\pi f l}{c}} + \frac{3}{13}e^{-j\frac{3\pi f l}{c}})$$

With $f = 60\,Hz$ and $l = 80\,m$,

$$V_L \approx 97.5(1 + \frac{3}{13}) = 120\angle 0^o\,V$$
$$\textit{Length of line << wavelength of voltage wave}$$

With $f = 1\,MHz$ and $l = 80\,m$,

$$V_L = 97.5(e^{j0.8\pi} + \frac{3}{13}e^{-j0.8\pi})$$
$$V_L = 97.5(-0.9957 + 0.4521j)$$
$$V_L = 106.622\angle 2.715^o\,V$$

With $f = 60\,Hz$ and $l = 10^7\,m$,

$$V_L = 97.5(e^{j6\pi} + \frac{3}{13}e^{-j6\pi})$$
$$V_L = 97.5(1 + \frac{3}{13}) = 120\angle 0^o\,V$$
$$\textit{Length of line multiple of wavelength of voltage wave}$$

With $f = 1\,MHz$ and $l = 10^7\,m$,

$$V_L = 97.5(e^{j10^5\pi} + \frac{3}{13}e^{-j10^5\pi})$$
$$V_L = 120\angle 0^o\,V$$
$$\textit{Length of line multiple of wavelength of voltage wave}$$