Theorem: The geometric multiplicity of an eigenvalue can not exceed its algebraic multiplicity[1].

Lemma: For a block matrix of the form $M = \begin{bmatrix} P & Q \\ 0 & R \end{bmatrix}$, the determinant $|M| = |P||R|$. This can be proved by performing row transformations on M to convert it to an upper triangular form $M' = \begin{bmatrix} P' & Q' \\ 0 & R' \end{bmatrix}$ (which leaves the determinant unchanged). The determinant of an upper triangular matrix is the product of the diagonal elements, which can be factored into the product of the diagonal elements of P' and Q', thus giving us $|M| = |M'| = |P'||Q'| = |P||Q|$. The last equality follows because row transformations would not have changed the determinants of P, Q. Note that it is important for 0 to have been in the lower left part of M. This way, the row transformations $R \to R'$ involve only the rows of R' (and not P), and hence leave the determinant of R unchanged.

Proof:

0. Setup: For an $n \times n$ matrix A, say that an eigenvalue λ has geometric multiplicity r i.e. there are r linearly independent eigenvectors, $\{v_i\}_{i=1}^r$, corresponding to this eigenvalue ($\gamma_A(\lambda) = r$).

1. Supplement these r eigenvectors with $s = n-r$ additional vectors, $\{w_i\}_{i=1}^s$ such that all the n vectors are all linearly independent. These n vectors are arranged as the columns of a matrix $S = [v_1 \ldots v_r \, w_1 \ldots w_s]$.

2. Consider the action of A on S: $AS = [\lambda v_1 \ldots \lambda v_r \, Aw_1 \ldots Aw_s]$. Since w_i need not be an eigenvector, there is no particular simplification in describing the action of A on it. However, since S specifies a basis for the entire space, we can say that Aw_i can be expressed as a linear combination of the basis vectors $\{v_i\}, \{w_i\}$. We write this as follows: $Aw_i = \sum_{j=1}^r v_j c_{ji} + \sum_{j=1}^s w_j d_{ji}$, which is easily generalized to: $A[w_1 \ldots w_s] = [v_1 \ldots v_r \, w_1 \ldots w_s] \begin{bmatrix} C \\ D \end{bmatrix}$, where $C \in \mathbb{C}^{r \times s}$ and $D \in \mathbb{C}^{s \times s}$.

3. Now construct the matrix $(A - tI)S$, where t is some scalar from the underlying field. We see the following factorization:

$$\begin{aligned} (A - tI)S &= [(\lambda - t)v_1 \ldots (\lambda - t)v_r \, (Aw_1 - tw_1) \ldots (Aw_s - tw_s)] \\ &= [v_1 \ldots v_r \, w_1 \ldots w_s] \begin{bmatrix} (\lambda - t)I_{r \times r} & C \\ 0_{s \times r} & D - tI_{s \times s} \end{bmatrix} \end{aligned}$$

where the elements of C, D are indicated in the action of A on w_i above (it is useful to have a column picture interpretation of the RHS).

4. We now use the earlier lemma, giving $|(A - tI)S| = |A - tI||S| = |S|(\lambda - t)^r |D - tI|$. Since S is composed of linearly independent vectors, $|S| \neq 0$, and it follows that

$$|A - tI| = (\lambda - t)^r p_s(t)$$

where $p_s(t) = |D - tI|$ is a polynomial in t of degree s.

5. Interpreting the above equation, we note that apart from the $(\lambda - t)^r$ factor, it is possible that λ is a root of $p_s(t)$ (and this will depend on D). Thus the algebraic multiplicity of λ (i.e. the number of times it is repeated) is *atleast* r, but can be more, i.e. $\mu_A(\lambda) \geq \gamma_A(\lambda)$. QED.

[1]Uday Khankhoje, Linear Algebra EE5120, July-Nov 2018, Electrical Engineering IIT Madras http://www.ee.iitm.ac.in/uday/2018b-EE5120/