EE5120 Linear Algebra: Tutorial 7, July-Dec 2018, Dr. Uday Khankhoje, EE IIT Madras

Covers 6.1,6.2,6.3 of GS

1. The following information is given about a 4×4 matrix A.

$$A[0.5, 0.5, , 0.5, 0.5]^T = 3[0.5, 0.5, 0.5, 0.5]^T,$$
$$A[0.5, -0.5i, -0.5, 0.5i]^T = 2.5[0.5, -0.5, 0.5, -0.5]^T,$$
$$A[0.5, -0.5, 0.5, -0.5]^T = [0.5, 0.5, -0.5, -0.5]^T,$$
$$A[0.5, 0.5i, -0.5, -0.5i]^T = 0.5[0.5, -0.5, -0.5, 0.5]^T,$$

Compute a rank i matrix A by considering only the first i equations from the above ($i = 1, 2, 3, 4$).

2. Consider a wireless communication system consisting of a transmitter (Tx) and a receiver (Rx). Tx wishes to transmit a data vector $\mathbf{x} \in \mathbb{C}^n$ to Rx. But Tx processes the data $\mathbf{x}$ before transmission. Thus, the final data which Tx transmits is $\tilde{\mathbf{x}} = P\mathbf{x}$, where $\tilde{\mathbf{x}}$ is the processed data and the $n \times n$ matrix P describes how Tx had processed the original data vector $\mathbf{x}$. The transmitted signal vector $\tilde{\mathbf{x}}$ passes through the wireless channel represented by the matrix H of rank r. The channel matrix H is known both to Tx and Rx. The signal vector received by Rx is given by,

$$\tilde{\mathbf{y}} = H\tilde{\mathbf{x}} + \tilde{\mathbf{w}},$$

where $\tilde{\mathbf{y}} \in \mathbb{C}^m$ is the observation vector and $\tilde{\mathbf{w}}$ is the low power additive unknown noise vector. Then Rx processes $\tilde{\mathbf{y}}$ and obtains a vector $\mathbf{y}$ as, $\mathbf{y} = Q\tilde{\mathbf{y}}$. It is given that $\mathbf{w} = Q\tilde{\mathbf{w}}$ will still be a low power unknown noise vector. Finally, Rx has to predict the information (or the original data vector $\mathbf{x}$) sent by Tx, for which Rx computes the least squares estimate of $\mathbf{x}$, denoted as $\hat{\mathbf{x}}_{\text{LS}}$. Now, answer the questions that follow:

(a) Suppose $r = n = m$ and H has n-dimensional eigen-space. Let $H = S\Lambda S^{-1}$ be the eigen-value decomposition of H.

(i) Write the expression for $\mathbf{y}$ in terms of $\mathbf{x}$ (the original data vector) and $\mathbf{w}$ (the processed noise vector), if $P = S$ and $Q = S^{-1}$.

(ii) Compute a simplified expression for $\hat{\mathbf{x}}_{\text{LS}}$. Observe that $\hat{\mathbf{x}}_{\text{LS}}(i) = a_i\mathbf{y}(i), \forall i = 1, ..., n$ where a_i's are some scalars, i.e., each element of $\mathbf{x}$ can be estimated independent of the other.

(iii) What are those a_i's?

[Note: For a vector $\mathbf{g}$, $\mathbf{g}(i)$ refers to i^{th} entry in the vector $\mathbf{g}$].

(b) Let $r < n \leq m$.

(i) Choose the matrices P and Q using SVD of H so that elements of $\mathbf{x}$ can be estimated independent of each other (just like the way in part (a)).

(ii) As done in (a), write an expression for $\mathbf{y}$ in terms of $\mathbf{x}$ and $\mathbf{w}$.

(iii) However, it is imperative to note here that you will not be able to estimate all the entries in $\mathbf{x}$. Guess why?

(iv) As a result of (iii), Tx will send only those many data elements in $\mathbf{x}$ that can be estimated, with remaining entires set to zero. Can you now say how many data points can be transmitted and estimated successfully in this case? Say the number of data points that can be sent is N. Can it be any N elements of $\mathbf{x}$?

(v) Having answered these, can you write a simplified design for matrices P and Q, i.e., fill only certain columns with what is needed and fill the remaining part of these matrices with zeros? Based on this design of P and Q, write the structure of the original data vector $\mathbf{x}$. Answer clearly.

(c) From the expression of $\mathbf{y}$ in part (b), the singular values of H, say $\left\{\sigma_k\right\}_{k=1}^{r}$, take the interpretation of the gain values offered by the channel H to the data elements present in $\mathbf{x}$. Suppose $0 \approx \sigma_i <<< \mathbf{w}(i)$, but clearly, $\sigma_i \neq 0, \forall i \in \mathcal{S}$, where $\mathcal{S} \subset \{1, ..., r\}$, and assuming that this information is known to Tx and Rx, what is the best way for the Tx to transmit data vectors so that Rx makes considerably a decent job of estimating them successfully with very less error.

(d) For this question, assume that Tx and Rx does not have the knowledge of H, but they know that rank of H is r. Both Tx and Rx obtain two estimates of H (somehow), say $\hat{H}_1$ and $\hat{H}_2$. Let $\left\{\alpha_k\right\}_{k=1}^{r}$ and $\left\{\beta_k\right\}_{k=1}^{r}$ be the sets of singular values of $\hat{H}_1$ and $\hat{H}_2$, such that $\alpha_1 = \beta_r = 0$. Let all the non-zero α_i's and β_i's be sufficiently large compared to the noise power level and $\alpha_i = \beta_i$, $\forall i = 2, ..., r-1$. Also assume that there is no error in estimating the unitary matrices that appears in the SVD of H. Again, the goal is to obtain estimates of entries (that can be possible) of $\mathbf{x}$ independent of each other (refer to condition in part (a) with scalars a_i's).

(i) How would you now precisely and compactly design matrices P and Q using the estimates **[case 1]** $\hat{H}_1$ alone, **[case 2]** $\hat{H}_2$ alone, and **[case 3]** $\hat{H}_1$ and $\hat{H}_2$ both?

(ii) Can the system successfully transmit and receive the same number of data points as found in part (b) in each of the above cases? Explain your answer clearly.

(iii) What would be the a_i's in part(a) here, for each of the above cases?

3. Suppose the factorization below is an SVD of a matrix A with the entries in U and V rounded to two decimal places.

$$A = \begin{bmatrix} 0.40 & -0.78 & 0.47 \\ 0.37 & -0.33 & -0.87 \\ -0.84 & -0.52 & -0.16 \end{bmatrix} \begin{bmatrix} 7.10 & 0 & 0 \\ 0 & 3.10 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.30 & -0.51 & -0.81 \\ 0.76 & -0.64 & -0.12 \\ 0.58 & -0.58 & 0.58 \end{bmatrix}$$

(a) What is the rank of A?

(b) Use this decomposition of A, with no calculations, to write a basis for $C(A)$, the column space of A, and a basis for $N(A)$, the null space of A.

(c) Repeat parts (a) and (b) for the matrix B

$$B = \begin{bmatrix} -0.86 & -0.11 & -0.50 \\ 0.31 & 0.68 & -0.67 \\ 0.41 & -0.73 & -0.55 \end{bmatrix} \begin{bmatrix} 12.48 & 0 & 0 & 0 \\ 0 & 6.34 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.66 & -0.03 & -0.35 & 0.66 \\ -0.13 & -0.90 & -0.39 & -0.13 \\ 0.65 & 0.08 & -0.16 & -0.73 \\ -0.34 & 0.42 & -0.84 & -0.08 \end{bmatrix}$$

4. A is an $m \times n$ matrix with singular value decomposition $A = U\Sigma V^T$, where U is an $m \times m$ orthonormal matrix,Σ is an $m \times n$ diagonal matrix with r positive entries, and V is an $n \times n$ orthonormal matrix. Justify the following

(a) Show that if A is square, then $|det(A)|$ is the product of the singular values of A.

(b) Show that if P is an orthonormal $m \times m$ matrix, then PA has the same singular values of A.

(c) Justify that the second singular value of a matrix A is the maximum of $||Ax||$ as x varies overall unit vectors orthonormal to v_1, with v_1 a right singular vector corresponding to first singular value of A.

5. Prove that a symmetric positive definite matrix has a unique symmetric positive definite square root.

6. The symbol $A \succ B$ means $A - B$ is positive definite, $A \prec B$ means $B - A$ is positve definite. Consider a symmetric positve definite matrix H.
   (a) Prove $mI \prec H \prec MI$ if and only if eigen values of H are bounded between m and M, where I is the identity matrix and $M > m > 0$.
   (b) Prove that the diagonal elements of H cannot be non-positive.
   (c) Given a symmetric matrix G, find appropriate m, M such that $mI \preceq G \preceq MI$.

7. Decide between a minimum, maximum, or saddle point for the following functions, if they are stationary points.
   (a) $F = -1 + 4(e^x - x) - 5x\sin(y) + 6y^2$ at the point $x = y = 0$.
   (b) $F = (x^2 - 2x)\cos(y)$, at the point $x = 1, y = \pi$.
   (c) $F = \frac{1}{4}x^4 + x^2y + y^2$ at the point $x = 1, y = 2$.

   Hint: Use the second derivative matrices.

8. (a) Solve the generalized eigenvalue problem i.e, $Ax = \lambda Bx$ for eigenvectors

$$\begin{bmatrix} 4 & 3 \\ 3 & 7 \end{bmatrix} x = \lambda \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} x$$

   Hint: Solve $|A - \lambda B| = 0$ for eigenvalues and substitute them in $Ax = \lambda Bx$ for eigenvectors. Also, If B is invertible. Then multiplying on both sides by B^{-1} gives $B^{-1}Ax = \lambda x$.

   (b) Solve the generalized eigenvector problem, i.e., $(A - \lambda I)^P x = 0$ for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

   Theory of generalized eigenvectors: $|A - \lambda I| = 0$ has to be solved for eigenvalues. For eigenvectors, you atmost need to solve for $(A - \lambda I)^k x = 0$, where k is the algebraic multiplicity of A. You start with P=1 and increment the P by 1, till you get all the eigenvectors.