EE5120 Linear Algebra: Tutorial 5, July-Dec 2018, Dr. Uday Khankhoje, EE IIT Madras
Covers Ch 4.1,4.2,5.1,5.2 of GS

1. Find the value of k in each of the following cases so that it satisfies the corresponding equation.

   (a)
   $$\det\begin{bmatrix} 3a & 3b & 3c \\ 3p & 3q & 3r \\ 3x & 3y & 3z \end{bmatrix} = k\det\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}.$$

   (b)
   $$\det\begin{bmatrix} 2a & 2b & 2c \\ 3p+5x & 3q+5y & 3r+5z \\ 7x & 7y & 7z \end{bmatrix} = k\det\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}.$$

   (c)
   $$\det\begin{bmatrix} p+x & q+y & r+z \\ a+x & b+y & c+z \\ a+p & b+q & c+r \end{bmatrix} = k\det\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}.$$

   Hint: Use properties of determinants.

Solution:

(a) Here we have,

$$\det\begin{bmatrix} 3a & 3b & 3c \\ 3p & 3q & 3r \\ 3x & 3y & 3z \end{bmatrix} = 3\det\begin{bmatrix} a & b & c \\ 3p & 3q & 3r \\ 3x & 3y & 3z \end{bmatrix} = 3^2\det\begin{bmatrix} a & b & c \\ p & q & r \\ 3x & 3y & 3z \end{bmatrix} = 3^3\det\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$$

Hence, value of k in this case is 27.

(b) Adding $-\frac{5}{7}$ times the third row to the second row, will not change the determinant. Thus, we get,

$$\det\begin{bmatrix} 2a & 2b & 2c \\ 3p+5x & 3q+5y & 3r+5z \\ 7x & 7y & 7z \end{bmatrix} = \det\begin{bmatrix} 2a & 2b & 2c \\ 3p & 3q & 3r \\ 7x & 7y & 7z \end{bmatrix}$$

Then, by proceeding the same way as part (a), we get $k = 2 \times 3 \times 7 = 42$.

(c) Here, we have,

$$\begin{aligned} \det\begin{bmatrix} p+x & q+y & r+z \\ a+x & b+y & c+z \\ a+p & b+q & c+r \end{bmatrix} &= -\det\begin{bmatrix} -(p+x) & -(q+y) & -(r+z) \\ a+x & b+y & c+z \\ a+p & b+q & c+r \end{bmatrix} \\ &= -2\det\begin{bmatrix} a & b & c \\ a+x & b+y & c+z \\ a+p & b+q & c+r \end{bmatrix} \\ &= -2\det\begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix} \\ &= 2\det\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}. \end{aligned}$$

In the above, the second step is obtaining by adding all second and third rows to first row. Third step is by subtracting first row from second and third row elements. And, the last step is got by interchanging rows two and three. Thus, the value of k, in this case, is 2.

2. Prove each of the following statements.

   (a) Two square matrices A and B, of same size, are said to be similar, if there exists an invertible matrix P of same size as that of A (or B) such that $A = PBP^{-1}$. Prove that determinant of A and B are same.

   (b) Let M be an $n \times n$ matrix with complex-valued entries in it. Matrix M^* refers to the complex conjugate of matrix M, i.e., if $[M]_{i,j}$ is the $(i,j)^{\text{th}}$ of matrix M, then $(i,j)^{\text{th}}$ element of the matrix M^* equals $[M]^*_{i,j}$. Show that det(M^*) = $\Big(\det(M)\Big)^*$.

   (c) Determinant of the matrix $A + tI$, where $t \neq 0$, I is an $n \times n$ identity matrix and,

$$A = \begin{bmatrix} 0 & 0 & 0 & ... & a_0 \\ -1 & 0 & 0 & ... & a_1 \\ 0 & -1 & 0 & ... & a_2 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & a_{n-1} \end{bmatrix},$$

   is equal to $t^n + \sum\limits_{i=0}^{n-1} a_i t^i$.

Hint: Use the properties of determinants for (a), and the definition of determinants for (b) and (c).

Solution:

(a) We have the following:

$$\det(A) = \det(P^{-1}BP) = \det(P^{-1})\det(B)\det P = \frac{1}{\det(P)}\det(B)\det(P) = \det(B).$$

Hence, proved.

(b) We prove by mathematical induction. First, let $n = 1$. Then, matrix $M = [M]_{1,1} \Rightarrow \det(M) = [M]_{1,1}$. And, $M^* = [M]^*_{1,1} \Rightarrow \det(M^*) = [M]^*_{1,1} = \Big(\det(M)\Big)^*$. Hence, it is true for $n = 1$. Let us assume that it is true for a (M, M^*) pair of matrices of size $n-1 \times n-1$. We now have to prove it for the size $n \times n$.

Suppose for a $k \times k$ matrix A, let $\tilde{A}_{i,j}$ be the sub-matrix of A obtained by considering all the elements of A in same order, except those elements that are present along i^{th} row and j^{th} column. Then, by the definition of determinants, we have,

$$\det(A) = \sum_{i=1}^{k}(-1)^{1+i}[A]_{1,i}\det(\tilde{A}_{1,i}). \tag{1}$$

From (1), we can write the following for a general $n \times n$ matrix M^*:

$$\begin{aligned}
\det(M^*) &= \sum_{i=1}^{n} (-1)^{1+i} [M]^*_{1,i} \det(\tilde{M}^*_{1,i}) \\
&= \sum_{i=1}^{n} \Big((-1)^{1+i}\Big)^* [M]^*_{1,i} \det(\tilde{M}^*_{1,i}), \qquad \Big[(-1)^{1+i} = \Big((-1)^{1+i}\Big)^*\Big] \\
&= \sum_{i=1}^{n} \Big((-1)^{1+i}\Big)^* [M]^*_{1,i} \Big(\det(\tilde{M}_{1,i})\Big)^*, \qquad \text{[By hypothesis assumed.]} \\
&= \Big\{ \sum_{i=1}^{n} (-1)^{1+i} [M]_{1,i} \det(\tilde{M}_{1,i}) \Big\}^* \\
&= \Big(\det(M)\Big)^*
\end{aligned}$$

Hence, proved.

(c) We have,

$$\begin{aligned}
\det \begin{bmatrix} t & 0 & 0 & \dots & a_0 \\ -1 & t & 0 & \dots & a_1 \\ 0 & -1 & t & \dots & a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n-1}+t \end{bmatrix} &= (t) \det \begin{bmatrix} t & 0 & 0 & \dots & a_1 \\ -1 & t & 0 & \dots & a_2 \\ 0 & -1 & t & \dots & a_3 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n-1}+t \end{bmatrix} \\
&\quad + a_0(-1)^{1+n} \det \begin{bmatrix} -1 & t & 0 & \dots & 0 \\ 0 & -1 & t & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 \end{bmatrix} \\
&= (t) \Bigg[(t) \det \begin{bmatrix} t & 0 & 0 & \dots & a_2 \\ -1 & t & 0 & \dots & a_3 \\ 0 & -1 & t & \dots & a_4 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n-1}+t \end{bmatrix} \\
&\quad + a_1(-1)^{2+n} \det \begin{bmatrix} -1 & t & 0 & \dots & 0 \\ 0 & -1 & t & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 \end{bmatrix} \Bigg] \\
&\quad + a_0(-1)^{1+n}(-1)^{n-1}
\end{aligned}$$

By expanding the determinants, we finally arrive at,

$$\det(A+tI) = t[a_1 + t[a_2 + t[\dots[a_{n-1}+t]..]] + a_0 = t^n + a_0 + a_1 t + a_2 t^2 + \dots + a_{n-1}t^{n-1}.$$

Hence, proved.

3. (a) Let $\mathsf{L} : \mathbb{M}_{n\times n} \to \mathbb{R}^{1\times n}$ be a linear transformation, with $\mathbb{M}_{n\times n}$ being the set of all $n \times n$ matrices, defined as $\mathsf{L}(P) = \mathbf{x}^T AP - \mathbf{x}^T P$, for any $P \in \mathbb{M}_{n\times n}$. Here, A is some fixed $n \times n$ symmetric matrix and $\mathbf{x}$ is some fixed $n \times 1$ column vector. If it is given that all invertible $n \times n$ matrices from $\mathbb{M}_{n\times n}$ map to $\mathbf{0}^T \in \mathbb{R}^{1\times n}$ under the transformation L,

can you comment about at least one eigenvalue and one eigenvector of matrix A?

(b) Let $H = I - 2\mathbf{u}\mathbf{u}^T$, where I is $n \times n$ identity matrix and $\mathbf{u}$ is an $n \times 1$ column vector such that $\mathbf{u}^T\mathbf{u} = 1$. Can you comment on at least two eigenvalues and corresponding eigenvectors of H?

(c) Let $A = \begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix}$, where $b \neq 0, c \neq 1$ and b, c are real numbers. Compute eigenvalues and eigenvectors of the matrices A and $B = \begin{bmatrix} A & O \\ O & A \end{bmatrix}$, where O is a 2×2 all-zero matrix.

Hint: Use definition of eigenvalue/eigenvector of a square matrix.

Solution:

(a) Given that all invertible matrices map to 0 under L. Let B be an invertible $n \times n$ matrix. Then,

$$\mathbf{x}^T AB - \mathbf{x}^T B = \mathbf{0}^T \Rightarrow B^T \left(A^T \mathbf{x} - \mathbf{x} \right) = \mathbf{0}. \tag{2}$$

Since, B is invertible, so is B^T. This implies all columns of B^T are linearly independent. Thus, from (2), we obtain, $A^T\mathbf{x} - \mathbf{x} = \mathbf{0} \Rightarrow A\mathbf{x} - \mathbf{x} = \mathbf{0}$ since A is symmetric. Finally, we get, $A\mathbf{x} = \mathbf{x}$. So, an eigenvalue of matrix A is equal to 1 and corresponding eigenvector is $\mathbf{x}$.

(b) Given: $H = I - 2\mathbf{u}\mathbf{u}^T$ with $\mathbf{u}^T\mathbf{u} = 1$. Now, multiplying by $\mathbf{u}$ on both sides, we get,

$$\begin{aligned} H\mathbf{u} &= \left(I - 2\mathbf{u}\mathbf{u}^T \right) \mathbf{u} \\ &= \mathbf{u} - 2\mathbf{u}\mathbf{u}^T\mathbf{u} = \mathbf{u} - 2\mathbf{u}(1) \\ &= -\mathbf{u}. \end{aligned}$$

Hence, $\mathbf{u}$ is an eigenvector of H with corresponding eigenvalue equal to -1.

Now, let $\mathbf{v}$ be any vector orthogonal to $\mathbf{u}$. Then,

$$\begin{aligned} H\mathbf{v} &= \left(I - 2\mathbf{u}\mathbf{u}^T \right) \mathbf{v} \\ &= \mathbf{v} - 2\mathbf{u}\mathbf{u}^T\mathbf{v} = \mathbf{v} - 2\mathbf{u}(0) \\ &= \mathbf{v}. \end{aligned}$$

Therefore, H has another eigenvalue equal to 1 with eigenvector as the one orthogonal to $\mathbf{u}$.

(c) Given A matrix is upper triangular. So, its eigenvalues are simply its diagonal entries $\Rightarrow \lambda_A = 1, c$. It can be verified that eigenvectors of A are $[1 \ 0]^T$, $[\frac{b}{c-1} \ 1]^T$.

B is a block diagonal matrix. In fact, since A is upper triangular, so is B. So, eigenvalues of B are: $1, 1, c, c$. Eigenvectors are given by, $[1 \ 0 \ 0 \ 0]^T$, $[0 \ 0 \ 1 \ 0]^T$, $[\frac{b}{c-1} \ 1 \ 0 \ 0]^T$, $[0 \ 0 \ \frac{b}{c-1} \ 1]^T$.

4. Show that the sum of eigenvalues of a matrix is given by its trace, and that the product of

eigenvalues is given by its determinant.

Solution:

(a) Let matrix A be of size $n \times n$. We know that $det(A - \lambda I)$ gives us the characteristic equation of the matrix, which can be written in terms of its eigenvalues as shown below,

$$\begin{aligned} det(A - \lambda I) &= (\lambda_1 - \lambda)(\lambda_2 - \lambda)...(\lambda_n - \lambda) \\ &= (-\lambda)^n + (-\lambda)^{n-1}(\lambda_1 + \lambda_2 + ... + \lambda_n) + ... + c \end{aligned}$$

And

$$det(A - \lambda I) = \begin{bmatrix} a_{11} - \lambda & a_{12} & .. & a_{1n} \\ a_{21} & a_{22} - \lambda & .. & a_{2n} \\ : & : & .. & : \\ : & : & .. & : \\ a_{n1} & . & .. & a_{nn} - \lambda \end{bmatrix}$$
$$= (-\lambda)^n + (-\lambda)^{n-1}(a_{11} + a_{22} + ... + a_{nn}) + ... + c$$

Equating λ^{n-1} terms in the above two simplifications, we get,

$$\lambda_1 + \lambda_2 + ... + \lambda_n = a_{11} + a_{22} + ... + a_{nn} = trace(A)$$

Hence proved.

(b) Let matrix A be of size $n \times n$. We know that $det(A - \lambda I)$ gives us the characteristic equation of the matrix, which can be written in terms of its eigenvalues as shown below,

$$det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)...(\lambda_n - \lambda)$$

Setting $\lambda = 0$

$$det(A - 0I) = (\lambda_1 - 0)(\lambda_2 - 0)...(\lambda_n - 0)$$
$$det(A) = \lambda_1\lambda_2...\lambda_n$$

Hence proved.

5. (i) Given that $A\mathbf{x} = \lambda\mathbf{x}$, prove the following:
   (a) λ^2 is an eigenvalue of A^2,
   (b) λ^{-1} is an eigenvalue of A^{-1},
   (c) $\lambda + 1$ is an eigenvalue of $A + I$.

   (ii) A 3×3 matrix B is known to have eigenvalues $0, 1, 2$. This information is enough to find three of these:
   (a) the rank of B,
   (b) the determinant of $B^T B$,
   (c) the eigenvalues of $B^T B$, and
   (d) the eigenvalues of $(B + I)^{-1}$.

Solution:

(i) Given, λ is eigenvalue of A matrix

(a) $A\mathbf{x} = \lambda\mathbf{x} \Rightarrow AA\mathbf{x} = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$.

(b) $A\mathbf{x} = \lambda\mathbf{x} \Rightarrow A^{-1}A\mathbf{x} = A^{-1}(\lambda\mathbf{x}) \Rightarrow \mathbf{x} = \lambda A^{-1}\mathbf{x} \Rightarrow \lambda^{-1}\mathbf{x} = A^{-1}\mathbf{x}$.

(c) $A\mathbf{x} = \lambda\mathbf{x} \Rightarrow A\mathbf{x} + \mathbf{x} = \lambda\mathbf{x} + \mathbf{x} \Rightarrow (A+I)\mathbf{x} = (\lambda+1)\mathbf{x}$.

(ii) Given $0, 1, 2$ as eigenvalues of $3\,by\,3$ matrix B.

(a) $rank(B) = 2$, i.e, rank is equal to the number of non-zero eigenvalues.

(b) $det(B^TB) = det(B^T)det(B) = det(B)det(B) = (0 \times 1 \times 2)^2 = 0$.

(c) Eigenvalues of B^T are same as eigenvalues of B. But, we can't find eigenvalues of B^TB from eigenvalues of B in general unless all eigenvectors of B are orthogonormal to each other.

(d) Eigenvalues of $B + I$ are $1, 2, 3$. So, the eigenvalues of $(B+I)^{-1}$ are $1, \frac{1}{2}, \frac{1}{3}$.

6. Prove that two $n \times n$ matrices are equal if all their eigenvalues and their corresponding eigenvectors are equal, and the matrices have n linearly independent eigenvectors.

Solution: Given: Matrices A and B have same eigenvalues and corresponding eigenvectors. Let λ_i and $\mathbf{v}_i$ be the i^{th} eigenvalue and corresponding eigenvector for $i \in 1, 2, ..., n$.

$$\begin{aligned} &A\mathbf{v}_i = B\mathbf{v}_i = \lambda_i\mathbf{v}_i, \quad \forall i \in 1, 2, ..., n \\ \Rightarrow &(A-B)\mathbf{v}_i = \mathbf{0}, \quad \forall i \in 1, 2, ..., n \\ \Rightarrow &(A-B)[\mathbf{v}_1\,\mathbf{v}_2\,...\,\mathbf{v}_n] = O, \end{aligned}$$

where O is an $n \times n$ all-zero matrix. Because the eigenvectors span $\mathbb{R}^n$ and all rows of $(A-B)$ are orthogonal to $\mathbb{R}^n$, the only possibility for the last equation (in the above) to hold true will be when all the rows of $(A-B)$ are zero vectors. Thus, we have,

$$A - B = O \Rightarrow A = B. \qquad \text{Hence, proved.}$$

7. The powers A^k of this matrix A approaches a limit as $k \to \infty$:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}, \quad A^2 = \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix}, \quad and \quad A^\infty = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$$

The matrix A^2 is halfway between A and A^∞. Explain why $A^2 = \frac{1}{2}(A + A^\infty)$ from the eigenvalues and eigenvectors of these three matrices. *Hint:* Use the fact from Q6.

Solution: Eigenvalues of $A : 1, 0.5$.
(If λ is eigenvalue of A, then λ^n is eigenvalue of A^n)
$\Rightarrow$ Eigenvalues of $A^2 : 1, 0.25$ and Eigenvalues of $A^\infty : 1, 0$.

We know eigenvectors are same for A, A^2, A^∞. Same for $\frac{1}{2}(A + A^\infty)$. It can be seen that eigenvalues of $\frac{1}{2}(A + A^\infty)$ equals $\frac{1}{2}[(1, 0.5) + (1, 0)] = (1, 0.25)$, which is equal to eigenvalues of A^2. Because eigenvalues and eigenvectors of A^2, $\frac{1}{2}(A + A^\infty)$ are equal. $\Rightarrow A^2 = \frac{1}{2}(A + A^\infty)$.

8. Consider a matrix A of size $n \times n$. If A has $(n_1 + 1)$ distinct eigen values and one of them is repeated n_2 number of times, satisfying $n_1 + n_2 = n$, then derive a condition that can ensure the diagonalizability of A.

Hint: Find the condition in terms of rank of some matrix.

Solution: Let $\hat{\lambda}$ be the eigen value that is repeated for n_2 number of times. For A to be diagonalizable, it should have n linearly independent eigen vectors. The set of eigen vectors correspondng to n_1 distict eigen values will be independent. The eigen vectors correponding to repeated eigen value are also need to be independent. The nullspace of the $n \times n$ matrix $A - \hat{\lambda}I$ are the eigen vectors corresponding to the eigen value $\hat{\lambda}$. Hence it's dimension should be n_2. Rank of $A - \hat{\lambda}I$ being $(n - n_2)$ will ensure this.

9. An $n \times n$ matrix M is said to be 'Markov matrix' if all it's entries are non-negative and the sum of the entries of each column is 1. If $\{\lambda_i\}$ are the eigen values of M and M is a real matrix, prove the followings
   (a) $\lambda_1 = 1$ is always an eigen value of M.
   (b) $|\lambda_i| \leq 1 \;\; \forall i \in \{1, \ldots n\}$.

Hint: (b) Use the fact that the matrices A and A^T have same eigen values.

Solution:

(a) $A = [a_{ij}]$ where $a_{ij} \geq 0$ and $\sum_{i=1}^{n} a_{ij} = 1 \;\; \forall j$
Consider the matrix $[A^T - I]$ and $n \times 1$ vector $\mathbf{x} = \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}^T$. Their product yields

$$\begin{aligned} \mathbf{y} &= [A^T - I]\mathbf{x} = A^T\mathbf{x} - \mathbf{x} \\ \implies y_j &= \sum_{i=1}^{n} a_{ij} - 1 = 0 \;\; \forall j \\ \implies \mathbf{y} &= \mathbf{0} \end{aligned}$$

$[A^T - 1I]$ has non-trivial nullspace, hence singular. Therefore, 1 is an eigen value of A^T. A and A^T have same eigen values. So, 1 is an eigen value of A.

(b) A and A^T have same eigen values. Let λ and $\mathbf{v}$ be the eigen value and the corresponding eigen vector of A^T. Let v_k be the entry of $\mathbf{v}$ that has maximum absolute value,
i.e., $|v_k| = \max(|v_1|, |v_2|, \ldots, |v_n|) > 0$ (equal to 0 will imply $\mathbf{v}$ itself is zero).

We have $A^T\mathbf{v} = \lambda\mathbf{v}$. Considering it's k^{th} row,

$$\begin{aligned}
\lambda v_k &= a_{1k}v_1 + a_{2k}v_2 + \cdots + a_{nk}v_n \\
|\lambda||v_k| &= |a_{1k}v_1 + a_{2k}v_2 + \cdots + a_{nk}v_n| \\
&\leq a_{1k}|v_1| + a_{2k}|v_2| + \cdots + a_{nk}|v_n| \quad \text{(Triangle inequality and } \ a_{ij} \geq 0) \\
&\leq a_{1k}|v_k| + a_{2k}|v_k| + \cdots + a_{nk}|v_k| \\
&= |v_k| \sum_{i=1}^{n} a_{ik} \\
&= |v_k| \\
\implies |\lambda| &\leq 1
\end{aligned}$$

Hence absolute value of all the eigen values of A^T, therefore A, will be less than or equal to 1.

10. If $p(\lambda) = \prod_{i=1}^{m}(\lambda - \lambda_i)$ is the characteristic polynomial of a matrix A with distinct eigen values, then find the characteristic polynomial of the matrix $A^n - kI$, where I is the identity matrix of appropriate dimension and $k, n \in \mathbb{R}$.

Solution: Let Λ be the diagonal matrix with eigen values of A in it's diagonal positions. Then there exists a matrix B such that $A = B\Lambda B^{-1}$. Further, $A^n = (B\Lambda B^{-1})^n = B\Lambda^n B^{-1}$.
The characteristic polynomial of $(A^n - kI)$ is given by

$$\begin{aligned}
\hat{p}(\lambda) = \det((A^n - kI) - \lambda I)) &= \det(B\Lambda^n B^{-1} - (k+\lambda)I) \\
&= \det(B\Lambda^n B^{-1} - (k+\lambda)BB^{-1}) \\
&= \det(B(\Lambda^n - (k+\lambda)I)B^{-1}) \\
&= \det(B)\det(\Lambda^n - (k+\lambda)I)\det(B^{-1}) \\
&= \det(\Lambda^n - (k+\lambda)I) \\
&= \prod_{i=1}^{m} \Big(\lambda_i^n - (k+\lambda)\Big)
\end{aligned}$$

11. Describe how you might try to build a solution of a difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$, $(k = 0, 1, 2, ...)$, if you were given the initial $\mathbf{x}_0$ and this vector did not happen to be an eigenvector of A. Assume A is an $p \times p$ matrix with all its p eigenvectors being linearly independent.

Solution: Let the eigenvectors of A be, $\mathbf{v}_1, ..., \mathbf{v}_p$, with eigenvalues $\lambda_1, ..., \lambda_p$. Since the eigenvectors are all independent and can span the $\mathbb{R}^p$ space, the initial solution vector $\mathbf{x}_0$ can be written as, $\mathbf{x}_0 = c_1\mathbf{v}_1 + ... + c_p\mathbf{v}_p$, i.e., as linear combination of the eigenvector. Then, we have the following:

$$\mathbf{x}_1 = A\mathbf{x}_0 = c_1\lambda_1\mathbf{v}_1 + ... + c_p\lambda_p\mathbf{v}_p.$$

By this way, we obtain,

$$\mathbf{x}_{k+1} = c_1\lambda_1^{k+1}\mathbf{v}_1 + ... + c_p\lambda_p^{k+1}\mathbf{v}_p.$$

In short, if $A = SDS^{-1}$ is the eigenvalue decomposition, then, $\mathbf{x}_{k+1} = A^{k+1}\mathbf{x}_0 = SD^{k+1}S^{-1}\mathbf{x}_0$.

12. Consider a linear operator T, $\mathsf{T} : \mathcal{V} \longrightarrow \mathcal{V}$, where $\mathcal{V}$ is a vector space and let $\mathcal{E}_\lambda = \{\mathbf{x} \Big| T(\mathbf{x}) = \lambda\mathbf{x}\}$ (called the eigen space of λ). Prove that $\mathcal{E}_\lambda$ is a subspace of $\mathcal{V}$.

Solution: We have to prove,
1) Zero vector is in $\mathcal{E}_\lambda$, which clearly holds true by definition of the set $\mathcal{E}_\lambda$.
2) If $\mathbf{u} \in \mathcal{E}_\lambda$, then $k\mathbf{u} \in \mathcal{E}_\lambda$ for any scalar k.
3) If $\mathbf{u}, \mathbf{v} \in \mathcal{E}_\lambda$, then $\mathbf{u} + \mathbf{v} \in \mathcal{E}_\lambda$.
Proceeding with the proof, we have,
i) $\mathbf{u} \in \mathcal{E}_\lambda \Rightarrow \mathsf{T}(\mathbf{u}) = \lambda\mathbf{u}$. Then, $\mathsf{T}(k\mathbf{u}) = k\mathsf{T}(\mathbf{u}) = k(\lambda\mathbf{u}) = \lambda\Big(k\mathbf{u}\Big)$.
So, $k\mathbf{u} \in \mathcal{E}_\lambda$.
ii) Let $\mathbf{u}, \mathbf{v} \in \mathcal{E}_\lambda$. Then,
$\mathsf{T}(\mathbf{u} + \mathbf{v}) = \mathsf{T}(\mathbf{u}) + \mathsf{T}(\mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v} = \lambda(\mathbf{u} + v\mathbf{v})$, so $\mathbf{u} + \mathbf{v} \in \mathcal{E}_\lambda$. Hence, proved.

13. An elastic object in the xy plane with a circular boundary $x^2 + y^2 = 1$ is stretched so that a point $\mathrm{P}(x_1, y_1)$ goes over into the point $\mathrm{Q}(x_2, y_2)$ given by

$$b = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \end{bmatrix}$$

Find the principal directions, that is the directions of the position vector d_1 of P for which the direction of the position vector d_2 of Q is the same or exactly opposite. What shape does the boundary circle take under the deformation?

Solution: Consider,

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

Eigenvalues of A are $\lambda = 8$ and 2.
Eigen vector for $\lambda_1 = 8$ is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and for $\lambda_2 = 2$ is,

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

These vectors make 45^0 and 135^0 angles with positive x direction. They give the principal directions. The eigen values show that in the principal directions, the object is stretched by factors 8 and 2, respectively as shown in fig 1.

To find the shape of deformation, choose the principal axes are the axes of new cartesian (u_1, u_2) coordinate system. Where u_1 is the semi-axis in the first quadrant and u_2 is the semi-axis in the second quadrant of the xy plane. Then, define
$u_1 = r\cos(\phi)$
$u_2 = r\sin(\phi)$
i.e., $\cos(\phi)$ and $\sin(\phi)$ are the coordinates of circular boundary (unstretched object). After stretching, we get,
$z_1 = 8\cos(\phi)$
$z_2 = 2\sin(\phi)$
and, $\cos^2(\phi) + \sin^2(\phi) = 1 \Rightarrow \frac{z_1^2}{8^2} + \frac{z_2^2}{2^2} = 1$. So the deformed object is an ellipse.

14. Let

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

(a) Find all eigenvalues and corresponding eigenvectors.

(b) Find a nonsingular matrix P such that $D = P^{-1}AP$ is diagonal, and P^{-1}.

(c) Find A^6 and $f(A)$, where $f(t) = t^4 - 3t^3 - 6t^2 + 7t + 3$.

(d) Find a real cube root of B, that is, a matrix B such that $B^3 = A$ and B has real eigenvalues. Assume B is diagonalizable.

Solution: (a) Eigenvalues are $\lambda = 1$ and 4. Eigenvector belonging to $\lambda = 1$ is $v_1 = (2, -1)$, and Eigenvector belonging to $\lambda = 4$ is $v_2 = (1, 1)$.
(b) Let P is the matrix whose columns are v_1 and v_2. Then

$$P = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$D = P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

(c) Using the diagonal factorization $A = PDP^{-1}$, and $1^6 = 1$ and $4^6 = 4096$, we get

$$A^6 = PD^6P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4096 \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 1366 & 2230 \\ 1365 & 2731 \end{bmatrix}$$

$f(1) = 2$ and $f(4) = -1$. So,

$$f(A) = Pf(D)P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$$

(d)

$$D^{1/3} = \begin{bmatrix} 1 & 0 \\ 0 & 4^{1/3} \end{bmatrix}$$

Hence the real cube root of A is

$$B = PD^{1/3}P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4^{1/3} \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 \\ 1/3 & 2/3 \end{bmatrix} = (1/3) \begin{bmatrix} 2+4^{1/3} & -2+2(4^{1/3}) \\ -1+4^{1/3} & 1+2(4^{1/3}) \end{bmatrix}$$