EE5120 Linear Algebra: Tutorial 3, July-Dec 2018, Dr. Uday Khankhoje, EE IIT Madras
Covers Ch 2.4, 2.6 of GS

1. (a) Find the dimension and a basis for the four fundamental subspaces for

$$(i)\ A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \quad (ii)\ U = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   (b) Without computing A, find bases for the four fundamental subspaces:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

   (c) Without multiplying matrices, find bases for the row and column spaces of A:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}$$

   How do you know from these shapes that A is not invertible?

   Hint:
   (a) Null spaces for A^T is left null space of A.
   (b) A is of the form LU and pivot elements are non-zero. So, Get $C(A)$, Left null spaces from L and $R(A)$, Null space from U.
   (c) See the number of pivot elements in both matrices, as calculate same as (b).

2. Given a rectangular matrix, prove that (a) if it has full row rank, then (AA^T) is invertible, (b) if it has full column rank, then (A^TA) is invertible. Finally, (c) prove that the left and right inverse of a square invertible matrix are identical.

   Hint: Think of the sizes of the null spaces of (A^TA) and (AA^T).

3. Given that a set of k vectors, $S_k = \{v_1, v_r, \ldots, v_k\}$ is linearly independent. Let us expand this set by adding a non-zero vector v_{k+1} that is orthogonal to all elements of S_k. Prove that the resulting set is also linearly independent.

   Hint: Use the definition of linear independence in matrix language.

4. (a) If A is square, (i) show that the nullspace of A^2 contains the nullspace of A. (ii) Show also that the column space of A^2 is contained in the column space of A.
   (b) If $AB = 0$, prove that $rank(A) + rank(B) \leq n$, where A is a $m \times n$ matrix.

5. Find the matrix representation for each of the following linear transformations. Also, say if the transformation is invertible or not just by looking at the matrix representation. Justify your answer.
   (a) Let $\mathbb{M}_2$ be the vector space of 2×2 real finite valued matrices, having an ordered basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. The linear transformation given is $\mathsf{L} : \mathbb{M}_2 \to \mathbb{R}$, and is defined as, $\mathsf{L}(A) = \text{trace}(A)$, where $A \in \mathbb{M}_2$.
   (b) Consider the linear transformation F on $\mathbb{R}^2$ defined by $\mathsf{F}(x, y) = (5x - y, 2x + y)$ and the following bases of $\mathbb{R}^2$: $\mathcal{E} = (e_1, e_2) = ((1, 0), (0, 1))$ and $\mathcal{S} = (u_1, u_2) = ((1, 4), (2, 7))$. Find the matrix A that represents F in the basis $\mathcal{E}$. Also, find the matrix B that represents F in the basis $\mathcal{S}$.

6. Let $\mathcal{V}$ be a vector space and $\mathsf{T} : \mathcal{V} \rightarrow \mathcal{V}$ be a linear transformation. Suppose $\mathbf{x} \in \mathcal{V}$ is such that $\mathsf{T}^k(\mathbf{x}) = \mathbf{0}$, $\mathsf{T}^m(\mathbf{x}) \neq \mathbf{0}$, $\forall \quad 1 \leq m < k$ and $k > 1$, then prove that the set of vectors $\{\mathbf{x}, \mathsf{T}(\mathbf{x}), \mathsf{T}^2(\mathbf{x}), ..., \mathsf{T}^{k-1}(\mathbf{x})\}$ is linearly independent.

7. Prove that,

   (a) A linear transformation $\mathsf{L} : \mathcal{V} \longrightarrow \mathcal{W}$ is invertible if and only if the matrix representation for L is square and its null-space has only all-zero element.

   (b) L^{-1} is also a linear transformation and $(\mathsf{L}^{-1})^{-1} = \mathsf{L}$.

8. Consider the problem, $A\mathbf{x} = \mathbf{b}$, with $\mathbf{b} = \begin{bmatrix} 4 \\ 6 \\ 10 \\ 14 \end{bmatrix}$. The set of all solutions is given by $\{\mathbf{x} \mid \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $c, d \in \mathbb{R}\}$

   (a) Find the size of the matrix A.

   (b) Find the dimension of all the four fundamental spaces of A.

   (c) Find the matrix A.

9. Matrix P is called a projector if $P^2 = P$. Suppose $\mathbf{v}$ is an n-length vector, with $\mathbf{v} \neq \mathbf{0}$ and $A = \frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}$.

   (a) Prove that A is a projector.

   (b) Let I be an $n \times n$ identity matrix. Are $I - A$, $I + A$ and $A(A^T A)^{-1} A^T$ projectors? Prove your answers. Assume $A^T A$ is invertible.

   (c) Let $\mathbf{v}_1 \neq \mathbf{0}$ be another n-length vector and $\mathbf{v}_2 = (I - A)\mathbf{v}_1$. Compute $\mathbf{v}^T\mathbf{v}_2$. What can you say about vectors $\mathbf{v}$ and $\mathbf{v}_2$?

Matlab Section (Optional)

Useful Matlab functions: `dftmtx`(N) $\rightarrow$ Generates $N \times N$ DFT matrix, `fft`$(\mathbf{x})$ $\rightarrow$ Generates the DFT of a vector $\mathbf{x}$.

1. Computing N-point DFT of a N-length sequence $\mathbf{x}$ is a linear transformation. Assuming $N = 4$, compute the matrix representation of this linear transformation using the standard basis (i.e. by giving one basis vector after the other to the `fft` command). Verify the obtained matrix with that generated using *dftmtx* command.

2. (a) Plot the point $(3, 0)$ in Matlab.

   (b) Generate a matrix that reflects $(3, 0)$ about the $x = y$ line and plot the resultant point in the same figure obtained in (a).

   (c) Compute the matrix that can project $(3, 0)$ onto the line $x = y$ and plot the resultant point in the same figure.

   (d) Evaluate the matrix which rotates the vector $(3, 0)$ by 60^0 clockwise and plot the final obtained vector too.

Code to visualize second problem:

```
%% Program to plot vector 'u' and its transformation vector 'v=Au'
%% 'u' is of size 2X1 and 'A' of size 2X2.
%% 'u','Ax' vector can be represented with (x,y) coordinates, in 2D-plane
clc;close all;clear all;
u = [1;2]; % Initalizing vector 'u'
A = [1 -1;-1 0]; % Initalizing matrix 'A'
v = A*u; % v is the transformed vector
figure(1);plot(u(1),u(2),'bs','MarkerSize',20, 'MarkerEdgeColor','blue',...
    'MarkerFaceColor',[0 0 1]);hold on;
plot(v(1),v(2),'rs','MarkerSize',20, 'MarkerEdgeColor','red',...
    'MarkerFaceColor',[1 0 0]);grid on;grid minor;
plot(0,0,'bo','MarkerSize',10, 'MarkerEdgeColor','black',...
    'MarkerFaceColor',[0 0 0]);
legend('vector "u"','Transformed vector "v"','Origin');
minn = min(min([u(1) v(1) u(2) v(2)])-2,-1);
maxx = max(max([u(1) v(1) u(2) v(2)])+2,1);
xlim([minn,maxx]);
ylim([minn,maxx]);
set(gca,'FontSize',30);
xlabel('x ---->','fontweight','bold','fontsize',30);
ylabel('y ---->','fontweight','bold','fontsize',30);
hTitle = title('Plot of points x, Ax');
set(hTitle,'FontSize',30); axis equal;
```