EE5120 Linear Algebra: Tutorial 2, July-Dec 2018, Dr. Uday Khankhoje, EE IIT Madras
Covers Ch 2.1,2.2,2.3 of GS

1. Is the product of lower triangular matrices always lower triangular?

Solution: Suppose A and B be $n \times n$ lower triangular matrices and define $C = AB$. Now, $(i,j)^{th}$ entry in matrix C can be written as, $C_{ij} = \sum_{k=1}^{N} A_{ik}B_{kj}$. Consider the following cases:

- $i = j$: Then, $C_{ii} = \sum_{k=1}^{N} A_{ik}B_{ki}$. Note that if $i < k$, $A_{ik} = 0$, and if $i > k, B_{ki} = 0$. Hence, the summation exists only when $i = k$, and we get,
$$C_{ii} = A_{ii}B_{ii}.$$
- $i < j$: If $k \leq i$, it implies $k < j$, then $B_{kj} = 0$. On the other hand, if $i < k \leq j$, $A_{ik} = 0$ for sure. Also, if $j < k \Rightarrow i < k$, we still have $A_{ik} = 0$. Thus, for every value of k, either $A_{ik} = 0$ or $B_{kj} = 0$ or both are zero. So, every product term $A_{ik}B_{kj}$ is equal to zero. Therefore, $C_{ij} = 0$ when $i < j$.

Thus, C is a lower triangular matrix.

2. Which of the following are sub-spaces of $\mathbb{R}^3$? Justify your answer.

(a) $\mathcal{V}_1 = \{(a_1, a_2, a_3) \Big| a_1 + a_2 + a_3 = 1\}$.

(b) $\mathcal{V}_2 = \{(b_1, b_2, b_3) \Big| b_2 = b_3, b_1 = 2b_2\}$.

(c) $\mathcal{V}_3 = \{(c_1, c_2, c_3) \Big| c_1 + 2c_2 + 3c_3 = 0\}$.

Solution:

(a) $\mathcal{V}_1$ is not a sub-space as $(0,0,0)$ will not be present in it.

(b) For the set $\mathcal{V}_2$, clearly $(0,0,0)$ is present. Now, say $(a_1, a_2, a_3), (c_1, c_2, c_3) \in \mathcal{V}_2$. So,
$$a_1 = 2a_2; a_2 = a_3; c_1 = 2c_2; c_2 = c_3. \tag{1}$$
Let $(d_1, d_2, d_3) = (a_1, a_2, a_3) + (c_1, c_2, c_3)$. Now, $d_2 = a_2 + c_2 = a_3 + c_3$ from equation (1). So, $d_2 = d_3$. At the same time, $d_1 = a_1 + c_1 = 2a_2 + 2c_2 = 2(a_2 + c_2) = 2d_2$. Thus, $d_1 = 2d_2$ and $d_2 = d_3$. Hence, $(d_1, d_2, d_3) \in \mathcal{V}_2$.

Suppose p is some scalar. Consider $(e_1, e_2, e_3) = p(a_1, a_2, a_3)$. Here, $e_1 = pa_1 = p(2a_2) = (p2)a_2 = (2p)a_2 = 2(pa_2) = 2e_2$ from equation (1). Also, $e_2 = pa_2 = pa_3 = e_3$. Thus, $(e_1, e_2, e_3) \in \mathcal{V}_2$. Hence, $\mathcal{V}_2$ is a sub-space.

(c) Again, it is evident that $(0,0,0) \in \mathcal{V}_3$. Let $(p_1, p_2, p_3), (q_1, q_2, q_3) \in \mathcal{V}_3$. Hence,
$$p_1 + 2p_2 + 3p_3 = 0; q_1 + 2q_2 + 3q_3 = 0. \tag{2}$$

Let $(d_1, d_2, d_3) = (p_1, p_2, p_3) + (q_1, q_2, q_3)$. Now,

$$\begin{aligned} d_1 + 2d_2 + 3d_3 &= p_1 + q_1 + 2(p_2 + q_2) + 3(p_3 + q_3) \\ &= p_1 + q_1 + 2p_2 + 2q_2 + 3p_3 + 3q_3 \\ &= (p_1 + 2p_2 + 3p_3) + (q_1 + 2q_2 + 3q_3) \\ &= 0 + 0 = 0. \end{aligned}$$

Thus, $(d_1, d_2, d_3) \in \mathcal{V}_3$. Now, let r be some scalar and $(e_1, e_2, e_3) = r(p_1, p_2, p_3)$. Here, $e_1 + 2e_2 + 3e_3 = rp_1 + 2rp_2 + 3rp_3 = r(p_1 + 2p_2 + 3p_3) = r.0 = 0 \Rightarrow (e_1, e_2, e_3) \in \mathcal{V}_3$. So, $\mathcal{V}_3$ is also a sub-space.

3. Let $\mathcal{W}$ be the set of all 2×2 matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

such that $A\mathbf{z} = 0$, where $\mathbf{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Is $\mathcal{W}$ a subspace of $\mathbb{M}_{22}$, where $\mathbb{M}_{22}$ is the vector space of all 2×2 real valued matrices? Explain.

Solution: $Az = 0 \implies Az = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Thus, $\mathcal{W}$ consists of all matrices of the form $\begin{bmatrix} a & -a \\ c & -c \end{bmatrix}$. Clearly, an all-zero matrix A will satisfy the equation $A\mathbf{z} = \mathbf{0}$ and hence, it will belong to $\mathcal{W}$. Now assume that $A_1 = \begin{bmatrix} a_1 & -a_1 \\ c_1 & -c_1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} a_2 & -a_2 \\ c_2 & -c_2 \end{bmatrix}$ are in $\mathcal{W}$. Then,

$$A_1 + A_2 = \begin{bmatrix} a_1 & -a_1 \\ c_1 & -c_1 \end{bmatrix} + \begin{bmatrix} a_2 & -a_2 \\ c_2 & -c_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & -(a_1 + a_2) \\ c_1 + c_2 & -(c_1 + c_2) \end{bmatrix}$$

is in $\mathcal{W}$. Consider a scalar k, then,

$$kA_1 = k \begin{bmatrix} a_1 & -a_1 \\ c_1 & -c_1 \end{bmatrix} = \begin{bmatrix} ka_1 & -(ka_1) \\ kc_1 & -(kc_1) \end{bmatrix}$$

is also in $\mathcal{W}$. Hence, $\mathcal{W}$ is a subspace of $\mathbb{M}_{22}$.

4. Suppose $\mathcal{V}$ is a vector space. Let $\mathcal{W}_1, \mathcal{W}_2 \subset \mathcal{V}$ be sub-spaces. Which of the following sets are sub-spaces? If a set is a sub-space, prove it. Else, provide a counter-example and state under what circumstance, it can be a sub-space.
   (a) $\mathcal{W}_1 \cap \mathcal{W}_2$.
   (b) $\mathcal{W}_1 \cup \mathcal{W}_2$.
   (c) $\mathcal{W}_3 = \{\mathbf{v} \big| \mathbf{v}^T\mathbf{u} = 0, \forall \mathbf{u} \in \mathcal{W}_1\}$.
   (d) $\mathcal{W} = \{\mathbf{w} \big| \exists \mathbf{w}_1 \in \mathcal{W}_1, \mathbf{w}_2 \in \mathcal{W}_2 \text{ satisfying } \mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2\}$.

Solution:

(a) Given that,

$$\mathcal{W}_1, \mathcal{W}_2 \text{ are sub-spaces.} \tag{3}$$

- Due to (3), $\mathbf{0} \in \mathcal{W}_1, \mathcal{W}_2$. Hence, $\mathbf{0} \in \mathcal{W}_1 \cap \mathcal{W}_2$.
- Let $\mathbf{u}, \mathbf{v} \in \mathcal{W}_1 \cap \mathcal{W}_2 \Rightarrow \mathbf{u}, \mathbf{v} \in \mathcal{W}_1$ and $\mathbf{u}, \mathbf{v} \in \mathcal{W}_2$. Due to (3), $\mathbf{u} + \mathbf{v} \in \mathcal{W}_1$ and $\mathbf{u} + \mathbf{v} \in \mathcal{W}_2 \Rightarrow \mathbf{u} + \mathbf{v} \in \mathcal{W}_1 \cap \mathcal{W}_2$.
- Suppose a is some scalar and $\mathbf{v} \in \mathcal{W}_1 \cap \mathcal{W}_2 \Rightarrow \mathbf{v} \in \mathcal{W}_1$ and $\mathbf{v} \in \mathcal{W}_2$. As a consequence of (3), $a\mathbf{v} \in \mathcal{W}_1$ and $a\mathbf{v} \in \mathcal{W}_2$. Hence, $a\mathbf{v} \in \mathcal{W}_1 \cap \mathcal{W}_2$.

Thus, $\mathcal{W}_1 \cap \mathcal{W}_2$ is a sub-space.

(b) In general, $\mathcal{W}_1 \cup \mathcal{W}_2$ is not a sub-space. Eg: Consider the vector space $\mathbb{R}^2$. The set of points defined by the line $x = 0$ is a sub-space. Similarly, the set of points defined by the set $y = 0$ is also a sub-space. But the union of these two sets is not a sub-space.

For any two sub-spaces, $\mathcal{W}_1$ and $\mathcal{W}_2$, $\mathcal{W}_1 \cup \mathcal{W}_2$ will be a sub-space only if one of the set is a subset of the other.

(c)
- Since $\mathbf{0}^T\mathbf{u} = 0$, $\forall \mathbf{u} \in \mathcal{W}_1$, $\mathbf{0} \in \mathcal{W}_3$.
- Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{W}_3$ and $\mathbf{u}$ be any vector in $\mathcal{W}_1$. Then, $\mathbf{v}_1^T\mathbf{u} = \mathbf{v}_2^T\mathbf{u} = 0$. Now, $\Big(\mathbf{v}_1 + \mathbf{v}_2\Big)^T\mathbf{u} = \Big(\mathbf{v}_1^T + \mathbf{v}_2^T\Big)\mathbf{u} = \mathbf{v}_1^T\mathbf{u} + \mathbf{v}_2^T\mathbf{u} = 0 + 0 = 0$. Thus, $\mathbf{v}_1 + \mathbf{v}_2 \in \mathcal{W}_3$.
- Suppose a is some scalar, $\mathbf{w} \in \mathcal{W}_3$ and $\mathbf{u}$ is any vector in $\mathcal{W}_1$. So, $\mathbf{w}^T\mathbf{u} = 0$. The, $\Big(a\mathbf{w}\Big)^T\mathbf{u} = a^T\mathbf{w}^T\mathbf{u} = a.0 = 0$. Hence, $a\mathbf{w} \in \mathcal{W}_3$. This is true for any scalar a and any vector in $\mathbf{u} \in \mathcal{W}_1$.

So, $\mathcal{W}_3$ is a sub-space.

(d)
- Since $\mathbf{0} \in \mathcal{W}_1, \mathcal{W}_2$, $\mathbf{0} + \mathbf{0} = \mathbf{0} \in \mathcal{W}$.
- Let $\mathbf{u}, \mathbf{v} \in \mathcal{W}$. Then, $\exists\Big\{\mathbf{u}_i, \mathbf{v}_i \in \mathcal{W}_i\Big\}_{i=1,2}$, s.t. $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ and $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. Now,

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{v}_1 + \mathbf{v}_2 \\ &= \Big(\mathbf{u}_1 + \mathbf{v}_1\Big) + \Big(\mathbf{u}_2 + \mathbf{v}_2\Big) = \mathbf{w}_1 + \mathbf{w}_2.\end{aligned}$$

  In the above, $\mathbf{u}_i + \mathbf{v}_i = \mathbf{w}_i \in \mathcal{W}_i$ since $\mathcal{W}_i$ is a sub-space for each $i = 1, 2$. Thus, $\mathbf{u} + \mathbf{v} \in \mathcal{W}$.
- Let $\mathbf{u} \in \mathcal{W}$ and a be any scalar. $\exists\Big\{\mathbf{u}_i \in \mathcal{W}_i\Big\}_{i=1,2}$ s.t. $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$. Now,

$$a\mathbf{u} = a\Big(\mathbf{u}_1 + \mathbf{u}_2\Big) = a\mathbf{u}_1 + a\mathbf{u}_2 = \mathbf{p}_1 + \mathbf{p}_2.$$

  Since, $\mathcal{W}_i$'s are sub-spaces, $\mathbf{p}_i = a\mathbf{u}_i \in \mathcal{W}_i$ for every $i = 1, 2$. Hence, $a\mathbf{u} \in \mathcal{W}$.

Thus, $\mathcal{W}$ is a sub-space.

5. (a) The span of the following set of vectors is a sub-space of ______ dimensional real space. Fill in the blank.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 3 \end{bmatrix}$$

(b) What is the dimension of the sub-space spanned by the vectors given in (a)?

(c) Following (a) and (b), Check whether the following statement is true.

Statement : If $\mathcal{S}$ is an m-dimensional vector space and $\mathcal{S} \subseteq \mathbb{R}^n$, n is always equal to m.

If true, justify/prove it. If false, specify the possible values that n can take.

(d) Find a set of vector(s) that yields 0 on taking inner product with any of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Comment whether the above computed set of vectors form a vector space. If it does, what is it's dimension? Compare this dimension with those obtained in (a) and (b).

Solution:

(a) 4. There are 4 entries in $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$.

(b) 1. Each of the three vectors is linearly dependent on every other vectors.

(c) No. $n \in \{m, m+1, m+2, \dots\}$

(d)
- $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ can be expressed as a linear combination of one basis vector as follows

$$a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \text{where } a \in \{1, 2, 3\}$$

- Hence, any vector that yields zero on taking inner product with $\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$ will yield the same with $\mathbf{v_1}$,$\mathbf{v_2}$,$\mathbf{v_3}$ and also with any of other vectors spanned by $\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$.
- Let $\mathcal{S} \subseteq \mathbb{R}^4$ be the required set of vectors
- Consider $\mathbf{x} \in \mathcal{S}$

$$\implies \mathbf{x}^T \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

$\implies x_1 = -x_4$

$\implies \mathbf{x}$ can be written as

$$\mathbf{x} = b \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \text{where } a, b, c \in \mathbb{R}$$

- We have expressed $\mathbf{x} \in \mathcal{S}$ as linear combination of three basis vectors. Hence $\mathcal{S}$ is a 3 dimensional subspace.
- Dimension obtained in (a) equals the addition of the dimensions obtained in (b) and (d). This will be proved when the concept of "Orthogonal spaces" is introduced.

6. Solve the following:
   (a) Reduce these matrices A and B to their ordinary echelon forms U:

$$(i)A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}, \quad (ii)B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}$$

   Find the special solution for each free variable and describe every solution to $Ax = 0$ and $Bx = 0$. Reduce the echelon forms U to R, and draw a box around the identity matrix in the pivot rows and pivot columns.

   (b) Find the column space and nullspace of A and the solution to $Ax = b$:

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

Solution:

(a)

$$(i)U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, R = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, Null\,space = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} \boxed{\begin{matrix} 1 \\ 0 \end{matrix}} & \begin{matrix} 2 \\ 0 \end{matrix} & \boxed{\begin{matrix} 0 \\ 1 \end{matrix}} & \begin{matrix} 0 & 0 \\ 2 & 3 \end{matrix} \\ 0 & 0 & 0 & 0 \;\; 0 \end{matrix}$$

$$(ii)U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, Null\,space = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{matrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 1 & -2 \\ 1 & 2 \end{matrix} \\ 0 \;\; 0 & 0 \;\; 0 \end{matrix}$$

(b)

$$R = \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, Null\,space = \begin{bmatrix} -1 & 2 \\ -1 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} Column\,space = \begin{bmatrix} 2 & 4 \\ 2 & 5 \\ 2 & 3 \end{bmatrix}$$

Solution for (Ax=b) $\quad x = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

7. R denotes the row-reduced echelon form of a 5x3 matrix A. R has three non-zero pivots.
   (a) Find the set of vector(s) that solve $R\mathbf{x} = \mathbf{0}$.
   (b) The matrix B is defined as $\begin{bmatrix} \text{R} \\ \text{2R} \end{bmatrix}$. Find the rank of B.
   (c) The matrix C is defined as $\begin{bmatrix} \text{R} & \text{R} \\ \text{R} & 0 \end{bmatrix}$. Find the rank of C.

Solution:

(a) $R\mathbf{x} = 0$
$\implies x_1R_1 + x_2R_2 + x_3R_3 = 0$
$\implies x_1 = x_2 = x_3 = 0$ (Since R has 3 pivots, it's columns are independent)
$\implies$ zero vector $\mathbf{x} = \mathbf{0}$ is the only solution to $R\mathbf{x} = \mathbf{0}$.

(b)

$$B = \begin{bmatrix} R \\ 2R \end{bmatrix} \simeq \begin{bmatrix} R \\ 0 \end{bmatrix}$$

The row reduced echelon form of B has 3 pivots. Hence, it's rank is 3.

(c)

$$C = \begin{bmatrix} R & R \\ R & 0 \end{bmatrix} \simeq \begin{bmatrix} R & R \\ 0 & -R \end{bmatrix} \simeq \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix} \simeq \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$$

There will be some zero rows in the middle of $\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}$, which can be swapped to get the row reduced echelon form of C. The row reduced echelon form of C will have 6 pivots, hence it's rank is 6.

8. Prove the following:
   (a) $Rank(AB) \leq Rank(A)$.
   (b) Suppose A and B are n by n matrices, and $AB = I$. Prove from $rank(AB) \leq rank(A)$ that the rank of A is n. So, A is invertible and B must be its inverse. Therefore, $BA = I$.

Solution:

(a) Let A and B be matrices of dimensions $m \times n$ and $n \times p$, respectively.

$$AB = \begin{bmatrix} Ab_1 & Ab_2 & .. & Ab_i & .. & Ab_p \end{bmatrix}$$

where $b_1, b_2, .., b_p$ are columns of B.

So, Columns of AB are linear combinations of columns of A. Rank(AB) can at most be equal to rank of A. i.e, $Rank(AB) \leq Rank(A)$.

(b) Given, $AB = I$ and A is invertible. Multiply A^{-1} on right side and A on left side of equation.

$$\Rightarrow A^{-1}(AB)A = A^{-1}(I)A$$

$$\Rightarrow BA = I$$

Hence proved.

9. (a) Which of the following sets of vectors are linearly independent? Justify your answer.
   - $\mathcal{S}_1 = \{(0,0,0,0)\}$.
   - $\mathcal{S}_2 = \{(1,1,1)\}$.
   - $\mathcal{S}_3 = \{(1,-1,0),(0,0,1),(1,1,0)\}$.

   (b) Suppose $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a set of finite number of m-length vectors ($m \geq n$). If $\mathbf{v}_j \neq \mathbf{0}$, $\forall j = 1, ..., n$ and $\mathbf{v}_i^T \mathbf{v}_k = 0$, $\forall i \neq k$ and $i, k = 1, ..., n$, then prove that $\mathcal{S}$ contains linearly independent vectors.

   (c) Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be sets containing finite number of vectors such that $\mathcal{S}_1 \subset \mathcal{S}_2$. Which of the following statements is/are True? Justify your answer. Prove the statement if it is true, else given a specific counter-example if the statement is false.
   - If $\mathcal{S}_2$ is linearly dependent, then so is $\mathcal{S}_1$.
   - If $\mathcal{S}_2$ is linearly independent, then so it $\mathcal{S}_1$.

Solution:

(a)
- $\mathcal{S}_1$ is a single-ton set containing all-zero vector. Since, $c(0,0,0,0) = (0,0,0,0)$ for any $c \neq 0$ too, given set $\mathcal{S}_1$ is not linearly independent.
- Again, $\mathcal{S}_2$ is a single-ton. Here, $c(1,1,1) = (0,0,0)$ only if $c = 0$. Hence, $\mathcal{S}_2$ is a linearly independent set. In fact, all singleton sets with a non-zero vector in it, are linearly independent.
- Let $a(1,-1,0) + b(0,0,1) + c(1,1,0) = (0,0,0)$. First, we obtain, $b = 0$. Then, $a + c = 0$ and $c - a = 0$, which implies, $a = c = 0$. So, the given set of vectors linearly combine to give an all-zero vector only when the vectors are weighted by zero co-efficients. Thus, $\mathcal{S}_3$ is linearly independent.

(b) Let $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_n\mathbf{v}_n = \mathbf{0}$, for some scalars $a_1, ..., a_n$. Now for some $k \in \{1, ..., n\}$, we have,

$$\begin{aligned} \mathbf{v}_k^T \left(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_n\mathbf{v}_n \right) &= \mathbf{v}_k^T \mathbf{0} \\ \Rightarrow a_k \mathbf{v}_k^T \mathbf{v}_k = 0. \end{aligned}$$

Since, $\mathbf{v}_j \neq 0$, $\forall j$, in the above equation $a_k = 0$. This is true for any $k \in \{1, ..., n\}$. Hence, $a_1 = a_2 = ... = a_n = 0$. Thus, $\mathcal{S}$ is linearly independent.

(c)
- False. Eg: $\mathcal{S}_2 = \{(1,1),(2,2)\}$ but $\mathcal{S}_1 = \{(1,1)\}$.
- True. Let $\mathcal{S}_2 = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ and $\mathcal{S}_1 = \{\mathbf{v}_1, ..., \mathbf{v}_k\}$, where $k < n < \infty$. Given $\mathcal{S}_2$ is linearly independent. **Assume that $\mathcal{S}_1$ is linearly dependent**. Hence, we have,

$$\sum_{i=1}^{k} a_i \mathbf{v}_i = \mathbf{0}, \tag{4}$$

with at least one of the a_i's being non-zero. This means in $\sum_{i=1}^{n} a_i \mathbf{v}_i$, even if $a_{k+1}, ..., a_n$ are all set to zero, we get,

$$\sum_{i=1}^{n} a_i \mathbf{v}_i = \sum_{i=1}^{k} a_i \mathbf{v}_i + \sum_{j=k+1}^{n} a_j \mathbf{v}_j = \sum_{i=1}^{k} a_i \mathbf{v}_i + 0 = \mathbf{0},$$

from equation (4), with at least one of the a_i's being non-zero. This contradicts the given fact that $\mathcal{S}_2$ is linearly independent. Hence, our assumption about $\mathcal{S}_1$ is wrong. Thus, $\mathcal{S}_1$ is linearly independent if $\mathcal{S}_2$ is so with $\mathcal{S}_1 \subset \mathcal{S}_2$.

10. (a) Find a basis for the given sub-spaces of $\mathbb{R}^3$ and $\mathbb{R}^4$.

    (i) All vectors of the form, $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where a=0.

    (ii) All vectors of the form, $\begin{bmatrix} a+c \\ a-b \\ b+c \\ -a+b \end{bmatrix}$.

    (iii) All vectors of the form, $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $a-b+5c=0$.

    (b) Let $\mathcal{V}$ be the vector space of 2×2 matrices, and $\mathcal{W}$ be the sub-space of symmetric matrices. Show that $\dim(\mathcal{W}) = 3$, by finding a basis of $\mathcal{W}$.

Solution:

(a) The desired basis vectors are:

(i) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$

(b) If a matrix A is symmetric, $A = A^T$, i.e., $a_{ij} = a_{ji} \implies A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$.

Taking $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, we claim that $\mathcal{S} = \{E_1, E_2, E_3\}$ is a basis of $\mathcal{W}$, i.e., (a) $\mathcal{S}$ spans $\mathcal{W}$ and (b) $\mathcal{S}$ is linearly independent.

(a) The above matrix $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = aE_1 + bE_2 + dE_3$, thus S spans W.

(b) Suppose $xE_1 + yE_2 + zE_3 = 0$, where x, y, z are unknown scalars. This can be written as,

$$x \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

This implies,

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and so, $x = 0, y = 0$ and $z = 0$. Thus $\mathcal{S}$ is linearly independent. Therefore, $\mathcal{S}$ is a basis of $\mathcal{W}$ and dim($\mathcal{W}$)$= 3$.

Matlab Section (Optional)

Useful Matlab Functions:
Reduced row echelon form: `rref`
Rank of the matrix: `rank`
Null space: `null`