EE5120 Linear Algebra: Tutorial 1, July-Dec 2018, Dr. Uday Khankhoje, EE IIT Madras

Covers Ch 1 of GS

1. Solve the following system of linear equations using Gaussian Elimination method, if the system has a solution.

   (a)
$$\begin{aligned} &x_1 + x_2 - x_3 + 2x_4 = 2,\\ &x_1 + x_2 + 2x_3 = 1,\\ &2x_1 + 2x_2 + x_3 + 2x_4 = 4. \end{aligned}$$

   (b)
$$\begin{aligned} &x_1 + x_2 + 3x_3 - x_4 = 0,\\ &x_1 + x_2 + x_3 + x_4 = 1,\\ &x_1 - 2x_2 + x_3 - x_4 = 1,\\ &4x_1 + x_2 + 8x_3 - x_4 = 0. \end{aligned}$$

2. Answer True or false. Justify by proving the statement if it is true, or give a specific counterexample if it is false.

   (a) If columns i^{th} and j^{th} of B are the same, so are columns i^{th} and j^{th} of AB.

   (b) If rows i^{th} and j^{th} of B are same, so are rows of i^{th} and j^{th} of AB.

   (c) If rows of i^{th} and j^{th} of A are same, so are rows i^{th} and j^{th} of AB.

   (d) $(AB)^2 = A^2B^2$

3. A, B, C are matrices of sizes m by n, n by p, p by q respectively

   (a) It is required to find the matrix ABC. It can be done in two ways viz. $(AB)C$ and $A(BC)$. Find the computational cost (number of separate multiplications) required for both the cases.

   (b) If A is 2 by 4, B is 4 by 7, and C is 7 by 10, which one will you prefer? $(AB)C$ or $A(BC)$?

   (c) Prove that $(AB)C$ is faster when $n^{-1} + q^{-1} < m^{-1} + p^{-1}$.

4. Suppose A is defined as,
$$A = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}.$$
   Derive all the conditions to be satisfied by the elements a, b, c, d, e and f of matrix A so that A can be decomposed as LL^T, where L is a lower triangular matrix with non-negative real finite values. Also, compute the matrix L.

5. Let A and B be $m \times n$ matrices, and $\tilde{A}$ and $\tilde{B}$ be $(m+1) \times (n+1)$ matrices defined as,
$$\tilde{A} = \begin{bmatrix} 1 & \mathbf{0}_1^T \\ \mathbf{0}_2 & A \end{bmatrix}; \tilde{B} = \begin{bmatrix} 1 & \mathbf{0}_1^T \\ \mathbf{0}_2 & B \end{bmatrix},$$
   where $\mathbf{0}_1$ is an $n \times 1$ all-zero vector and $\mathbf{0}_2$ is an $m \times 1$ all-zero vector. Prove that if A can be transformed into B by an elementary row (or column) operation, then so can $\tilde{A}$ to $\tilde{B}$.

6. (a) Let W and Y be invertible matrices of size $n \times n$ and $m \times m$ respectively. Suppose X and Z are matrices of dimensions $n \times m$ and $m \times n$ respectively, then prove that

$$(W + XYZ)^{-1} = W^{-1} - W^{-1}X\left(Y^{-1} + ZW^{-1}X\right)^{-1}ZW^{-1},$$

provided $\left(Y^{-1} + ZW^{-1}X\right) \neq O$.

(b) Using result from (a), deduce the condition for which inverses of each of the following matrices exists. Also, find the inverse.

(i) $A + B$, where A and B are $n \times n$ invertible matrices.

(ii) $I + \mathbf{u}\mathbf{v}^T$, where $\mathbf{u}$ and $\mathbf{v}$ are n-length vectors, I is an $n \times n$ identity matrix.

Hint: Use definition of matrix inverse given in Ch 1.

7. State whether the following statement are true or false and justify. "A linear system with (at least) two solutions has infinitely many solutions".

8. (a) Prove that the LU decomposition of an invertible matrix is unique.

(b) Without computing A or A^{-1} or A^{-2} or A^2 explicitly, compute $A^{-1}x + A^{-2}y$, where you are given the following LU factorization A=LU:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, y = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

9. Express the matrix product AB^T as a sum of outerproducts of vectors.

Matlab Section (Optional)

Basics

MATLAB for beginners - Basic Introduction:
`https://www.youtube.com/watch?v=vF7cSmS83WU`
How to Write a MATLAB Program:
`https://www.youtube.com/watch?v=zr_aB7V79DE`

Problems:

(a) Write a MATLAB code to do LU decomposition (Input: Matrix A, Output: L,U).

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Algorithm without pivoting(LU = A):
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Initialize U=A, L=I
for k= 1:m-1
    for j=k+1:m
        L(j,k) =U(j,k)/U(k,k)
        U(j,1:m) =U(j,1:m)-L(j,k)U(k,1:m)
    end
end
```

Reference: `http://www.math.iit.edu/~fass/477577_Chapter_7.pdf`

(b) Find the computational complexity for solving $Ax = b$, by using a randomly generated matrix 'A' and a column vector 'b' of sizes $n \times n$ and $n \times 1$ (use in-built matlab function 'rand'). To find this, plot the computation time (by enclosing the relevant commands between the matlab commands 'tic' and 'toc') as a function of 'n' and fit a polynomial in 'n' to this curve. Compare with the matlab runtime for the same problem by using 'x = A\b' to solve the same problem.

(c) Using L,U matrices obtained from part(a) (OR) use inbuilt matlab function 'lu' and solve Ax=b. Compare the computation time with part(b).

(In many practical applications, matrix 'A' is fixed and 'b' will only vary. In those kind of problems it is good to do LU decomposition once and then solve for 'x' using L,U.)