EE5120 Linear Algebra: Tutorial 1, July-Dec 2018, Dr. Uday Khankhoje, EE IIT Madras
Covers Ch 1 of GS

1. Solve the following system of linear equations using Gaussian Elimination method, if the system has a solution.

   (a)

$$x_1 + x_2 - x_3 + 2x_4 = 2,$$
$$x_1 + x_2 + 2x_3 = 1,$$
$$2x_1 + 2x_2 + x_3 + 2x_4 = 4.$$

   (b)

$$x_1 + x_2 + 3x_3 - x_4 = 0,$$
$$x_1 + x_2 + x_3 + x_4 = 1,$$
$$x_1 - 2x_2 + x_3 - x_4 = 1,$$
$$4x_1 + x_2 + 8x_3 - x_4 = 0.$$

Solution:

(a) **Step 1:** R2 → R2 - R1; R3 → R3 - R1. We get,

$$x_1 + x_2 - x_3 + 2x_4 = 2; 3x_3 - 2x_4 = -1; 3x_3 - 2x_4 = 0.$$

Step 2: R3 → R3-R2. We obtain,

$$x_1 + x_2 - x_3 + 2x_4 = 2; 3x_3 - 2x_4 = -1; 0 + 0 = 1.$$

Hence, the given system of linear equations has no solution.

(b) **Step 1:** R2 → R2 - R1; R3 → R3 - R1; R4 → R4 - 4R1. We get,

$$x_1 + x_2 + 3x_3 - x_4 = 0; -2x_3 + 2x_4 = 1; -3x_2 - 2x_3 = 1; -3x_2 - 4x_3 + 3x_4 = 0.$$

Step 2: R4 → R4 - R3. As a consequence,

$$x_1 + x_2 + 3x_3 - x_4 = 0; -2x_3 + 2x_4 = 1; -3x_2 - 2x_3 = 1; -2x_3 + 3x_4 = -1.$$

Step 3: R4 → R4 - R2. Here, we get,

$$x_1 + x_2 + 3x_3 - x_4 = 0; -2x_3 + 2x_4 = 1; -3x_2 - 2x_3 = 1; x_4 = -2.$$

Thus, we get $x_4 = -2 \Rightarrow x_3 = -2.5 \Rightarrow x_2 = \frac{4}{3}$ and finally, $x_1 = \frac{25}{6}$.

2. Answer True or false. Justify by proving the statement if it is true, or give a specific counterexample if it is false.

   (a) If columns i^{th} and j^{th} of B are the same, so are columns i^{th} and j^{th} of AB.

   (b) If rows i^{th} and j^{th} of B are same, so are rows of i^{th} and j^{th} of AB.

   (c) If rows of i^{th} and j^{th} of A are same, so are rows i^{th} and j^{th} of AB.

(d) $(AB)^2 = A^2B^2$

Solution:

(a) True

(b) False

(c) True

(d) False, $(AB)^2 = ABAB \neq A^2B^2$

3. A, B, C are matrices of sizes m by n, n by p, p by q respectively

   (a) It is required to find the matrix ABC. It can be done in two ways viz. $(AB)C$ and $A(BC)$. Find the computational cost (number of separate multiplications) required for both the cases.

   (b) If A is 2 by 4, B is 4 by 7, and C is 7 by 10, which one will you prefer? $(AB)C$ or $A(BC)$?

   (c) Prove that $(AB)C$ is faster when $n^{-1} + q^{-1} < m^{-1} + p^{-1}$.

Solution:

(a) $(AB)C$ requires $mnp + mpq$ multiplications.
$A(BC)$ requires $mnq + npq$ multiplications.

(b) $(AB)C$ requires $(2)(4)(7) + (2)(7)(10) = 196$ multiplications
$A(BC)$ requires $(2)(4)(10) + (4)(7)(10) = 360$ multiplications.
Hence, $(AB)C$ is better than $A(BC)$

(c) For $(AB)C$ to be faster than $A(BC)$, we need $mnp + mpq < mnq + npq$.
Dividing both sides by $mnpq$, we get $q^{-1} + n^{-1} < p^{-1} + m^{-1}$

4. Suppose A is defined as,

$$A = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}.$$

Derive all the conditions to be satisfied by the elements a, b, c, d, e and f of matrix A so that A can be decomposed as LL^T, where L is a lower triangular matrix with non-negative real finite values. Also, compute the matrix L.

Solution: Let matrix L be
$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$. Then, LL^T is given by,

$$LL^T = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{21}l_{31} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}.$$

Equating LL^T to A, we get,

$$\begin{aligned}
l_{11} &= \sqrt{a}, \\
l_{21} &= \frac{d}{\sqrt{a}}, \\
l_{22} &= \sqrt{b - \frac{d^2}{a}}, \\
l_{31} &= \frac{e}{\sqrt{a}}, \\
l_{32} &= \frac{f - \frac{de}{a}}{\sqrt{b - \frac{d^2}{a}}}, \\
l_{33} &= \sqrt{c - \frac{e^2}{a} - \frac{(f - \frac{de}{a})^2}{b - \frac{d^2}{a}}}.
\end{aligned}$$

To ensure the elements of L are non-negative, real and finite, the required conditions are:

$$a > 0; d, e \geq 0; ab > d^2; af \geq ed; abc + 2def \geq af^2 + be^2 + cd^2.$$

5. Let A and B be $m \times n$ matrices, and $\tilde{A}$ and $\tilde{B}$ be $(m+1) \times (n+1)$ matrices defined as,

$$\tilde{A} = \begin{bmatrix} 1 & \mathbf{0}_1^T \\ \mathbf{0}_2 & A \end{bmatrix}; \tilde{B} = \begin{bmatrix} 1 & \mathbf{0}_1^T \\ \mathbf{0}_2 & B \end{bmatrix},$$

where $\mathbf{0}_1$ is an $n \times 1$ all-zero vector and $\mathbf{0}_2$ is an $m \times 1$ all-zero vector. Prove that if A can be transformed into B by an elementary row (or column) operation, then so can $\tilde{A}$ to $\tilde{B}$.

Solution: Suppose E is the elementary row transformation matrix such that, $B = EA$. Then, it can be verified that $\tilde{B} = \tilde{E}\tilde{A}$, with $\tilde{E}$ being,

$$\tilde{E} = \begin{bmatrix} 1 & \mathbf{0}_2^T \\ \mathbf{0}_2 & E \end{bmatrix},$$

which is clearly an elementary row transformation matrix. The same holds true for column transformation too.

6. (a) Let W and Y be invertible matrices of size $n \times n$ and $m \times m$ respectively. Suppose X and Z are matrices of dimensions $n \times m$ and $m \times n$ respectively, then prove that

$$(W + XYZ)^{-1} = W^{-1} - W^{-1}X\left(Y^{-1} + ZW^{-1}X\right)^{-1}ZW^{-1},$$

provided $\left(Y^{-1} + ZW^{-1}X\right) \neq O$.

(b) Using result from (a), deduce the condition for which inverses of each of the following matrices exists. Also, find the inverse.

(i) $A + B$, where A and B are $n \times n$ invertible matrices.
(ii) $I + \mathbf{u}\mathbf{v}^T$, where $\mathbf{u}$ and $\mathbf{v}$ are n-length vectors, I is an $n \times n$ identity matrix.

Hint: Use definition of matrix inverse given in Ch 1.

Solution:

(a) We have the following:

$$\begin{aligned}
&(W{+}XYZ)\Big(W^{-1} - W^{-1}X\Big[Y^{-1} + ZW^{-1}X\Big]^{-1}ZW^{-1}\Big) \\
&\quad = I + XYZW^{-1} - X\Big[Y^{-1} + ZW^{-1}X\Big]^{-1}ZW^{-1} - XYZW^{-1}X\Big[Y^{-1} + ZW^{-1}X\Big]^{-1}ZW^{-1} \\
&\quad = I + XYZW^{-1} - \Big(X + XYZW^{-1}X\Big)\Big[Y^{-1} + ZW^{-1}X\Big]^{-1}ZW^{-1} \\
&\quad = I + XYZW^{-1} - XY\Big(Y^{-1} + ZW^{-1}X\Big)\Big[Y^{-1} + ZW^{-1}X\Big]^{-1}ZW^{-1} \\
&\quad = I + XYZW^{-1} - XY\Big(I\Big)ZW^{-1} \\
&\quad = I.
\end{aligned}$$

Similarly, it can be verified that $\Big(W^{-1} - W^{-1}X\Big[Y^{-1} + ZW^{-1}X\Big]^{-1}ZW^{-1}\Big)(W + XYZ) = I$.

(b) (i) Matrix can be viewed as $A + IBI$, where I is an $n \times n$ identity matrix. Thus, inverse can be written as, $A^{-1}\Big(B^{-1} + A^{-1}\Big)^{-1}B^{-1}$ (after simplification). It can also be $B^{-1}\Big(B^{-1} + A^{-1}\Big)^{-1}A^{-1}$ as the given matrix can be viewed as $B + IAI$. For the inverse to exist the condition to be satisfied is: $A^{-1} \neq -B^{-1}$.

(ii) Here, the given matrix can be viewed as $I + \mathbf{u}(1)\mathbf{v}^T$. Hence, inverse is given by, $I - \frac{\mathbf{u}\mathbf{v}^T}{1+\mathbf{v}^T\mathbf{u}}$. For this inverse to exist, the required condition to be met is: $\mathbf{v}^T\mathbf{u} \neq -1$.

7. State whether the following statement are true or false and justify. "A linear system with (at least) two solutions has infinitely many solutions".

Solution: True. Suppose $(x_1, ...x_n)$ and $(y_1, ..., y_n)$ are two distinct solution of a linear system. Then the i^{th} row of the system can be represented as

$$a_{i1}x_1 + ... + a_{in}x_n = b_i$$

and

$$a_{i1}y_1 + ... + a_{in}y_n = b_i$$

we have to prove that $(px_1 + (1 - py_1), ..., px_n + (1 - py_n))$ is also a solution of the system. ie, if x and y are the two points in a line and $p \in [0, 1]$, then the equation $px + (1 - p)y$ represents the line segment between x and y ($x \neq y$). If we multiply the first equation by p, multiply the second equation by $(1 - p)$ and and add the two equations, we get

$$(pa_{i1}x_1 + ... + pa_{in}x_n) + ((1-p)a_{i1}y_1 + ... + (1-p)a_{in}y_n) = pb_i + (1-p)b_i$$
$$a_{i1}(px_1 + (1-p)y_1) + ... + a_{in}(px_n + (1-p)y_n) = b_i$$

Then, $(px_1 + (1-py_1), ..., px_n + (1-py_n))$ is also a solution. If we let p vary, we get infinitely many solutions.

8. (a) Prove that the LU decomposition of an invertible matrix is unique.
   (b) Without computing A or A^{-1} or A^{-2} or A^2 explicitly, compute $A^{-1}x + A^{-2}y$, where you are given the following LU factorization A=LU:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, y = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution:

(a) Assumption: LU decomposition of a matrix is not unique.
Then, consider the two distinct LU decomposition for A

$$A = L_1U_1$$

and

$$A = L_2U_2$$

A is invertible $\implies$ det(A)$\neq$0
$\implies$det(A) = det(L_1)det(U_1) = det(L_2)det(U_2)
$\implies$ det(L_1), det(U_1), det(L_2), det(U_1) $\neq 0$ i.e., L_1, U_1, L_2 and U_2 are invertible. So,

$$L_1U_1 = L_2U_2 \implies L_2^{-1}L_1 = U_2U_1^{-1} \quad (1)$$

LHS of Equ.1 is lower triangular matrix and RHS is upper triangular. This is only true for diagonal matrices. Also, the diagonal entries are 1.

$$\implies L_2^{-1}L_1 = I = U_2U_1^{-1}$$

$$\implies L_1 = L_2 \text{ and } U_1 = U_2.$$

So, the assumption is wrong and LU decomposition of an invertible matrix is unique.

(b)

$$A^{-1}x + A^{-2}y = A^{-1}(x + A^{-1}y)$$

To find $A^{-1}y$, consider,

$$Ap = y$$

$$\implies LUp = y$$

$$\implies Lq = y$$

where q=Up. Matrix L and y are given. Then,

$$q = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}.$$

Then, $p = A^{-1}y$=$U^{-1}q$

$$p = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \implies A^{-1}y = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \text{ and } x + A^{-1}y = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}.$$

Similarly, now solve $Lr = x + A^{-}y$ and Us=r. Then, we will get,

$$r = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \text{ and } s = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}.$$

Hence,

$$A^{-1}x + A^{-2}y = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}.$$

9. Express the matrix product AB^T as a sum of outerproducts of vectors.

Solution: Recall the defn of the outer product of two vectors p, q is $(pq^T)_{i,j} = p_i q_j$.

$$AB^T = \begin{bmatrix} A_1 A_2 ....A_n \end{bmatrix} \begin{bmatrix} B_1^T \\ B_2^T \\ . \\ . \\ . \\ . \\ B_n^T \end{bmatrix} = A_1 B_1^T + A_2 B_2^T + ... + A_n B_n^T$$

where A_i, B_i are the columns of the matrix A, B, respectively. The usual definition of matrix multiplication can be applied to the LHS of the above eqn and the defn of outer product to the RHS of this equation to see the equality.

Matlab Section (Optional)

Basics

MATLAB for beginners - Basic Introduction:
https://www.youtube.com/watch?v=vF7cSmS83WU
How to Write a MATLAB Program:
https://www.youtube.com/watch?v=zr_aB7V79DE

Problems:

(a) Write a MATLAB code to do LU decomposition (Input: Matrix A, Output: L,U).

```
Algorithm without pivoting(LU = A):
-------
Initialize U=A, L=I
for k= 1:m-1
    for j=k+1:m
        L(j,k) =U(j,k)/U(k,k)
        U(j,1:m) =U(j,1:m)-L(j,k)U(k,1:m)
    end
end
```

Reference: `http://www.math.iit.edu/~fass/477577_Chapter_7.pdf`

(b) Find the computational complexity for solving $Ax = b$, by using a randomly generated matrix 'A' and a column vector 'b' of sizes $n \times n$ and $n \times 1$ (use in-built matlab function 'rand'). To find this, plot the computation time (by enclosing the relevant commands between the matlab commands 'tic' and 'toc') as a function of 'n' and fit a polynomial in 'n' to this curve. Compare with the matlab runtime for the same problem by using 'x = A\b' to solve the same problem.

(c) Using L,U matrices obtained from part(a) (OR) use inbuilt matlab function 'lu' and solve Ax=b. Compare the computation time with part(b).

(In many practical applications, matrix 'A' is fixed and 'b' will only vary. In those kind of problems it is good to do LU decomposition once and then solve for 'x' using L,U.)