ELL212 - Tutorial 7, Sem II 2015-16

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Problems on wave-propagation:

1) **Poynting vector for monochromatic waves:** Consider a monochromatic electromagnetic field (frequency ω) with electric field $\mathbf{E}(\mathbf{r},t) = \mathbf{A}_e(\mathbf{r})\cos(\omega t - \phi_e(\mathbf{r}))$ and magnetic field $\mathbf{H}(\mathbf{r},t) = \mathbf{A}_h(\mathbf{r})\cos(\omega t - \phi_h(\mathbf{r}))$.
   a) Write down expressions for the complex electric and magnetic fields ($\tilde{\mathbf{E}}(\mathbf{r})$ and $\tilde{\mathbf{H}}(\mathbf{r})$) corresponding to the *real* electromagnetic fields mentioned above.
   b) Starting from the expression for the instantaneous Poynting vector $\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t)$, show that the time averaged Poynting vector is given by:

$$\tilde{\mathbf{S}}(\mathbf{r}) = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \mathbf{S}(\mathbf{r},t)\ \mathrm{d}t = \frac{1}{2}\mathrm{Re}(\tilde{\mathbf{E}}(\mathbf{r}) \times \tilde{\mathbf{H}}^*(\mathbf{r})) \tag{1}$$

2) **Power flow in standing waves:** A *standing electromagnetic wave* is formed by the superposition of electromagnetic of equal amplitude travelling in both forward and backward directions. Consider a 1D setup in vacuum with a forward propagating electromagnetic wave $\mathbf{E}_f(x) = \hat{z}E_0\exp(-ik_0x)$ and a backward propagating electromagnetic wave $\mathbf{E}_b(x) = \hat{z}E_0\exp(ik_0x - i\phi_0)$ (Note that the electric fields are complex fields with an implicit time dependence of $\exp(i\omega t)$ which has not been explicitly shown).
   c) Compute the net *real* electric and *real* magnetic fields in space.
   d) Use the result of Problem 1 to compute the net *time-averaged* Poynting vector in space. Also give a physical interpretation of your answer.

3) **Electromagnetic energy in conductors:** Consider a plane monochromatic wave at frequency ω propagating along $+x$ direction in a non-magnetic medium with relative permittivity ϵ and conductivity σ. Compute the ratio of the time-average electrical energy per unit volume to the time-average magnetic energy per unit volume. Which form of energy dominates in a good conductor? Note that the instantaneous electrical energy per unit volume is given by $u_E = \mathbf{D}\cdot\mathbf{E}/2$ and the instantaneous magnetic energy per unit volume is given by $u_H = \mathbf{H}\cdot\mathbf{B}/2$, where all the vectors correspond to the *real* fields in space.

4) **Propagation of non-monochromatic waves:** Consider a medium which has the following relationship between the propagation constant k and the frequency ω:

$$k(\omega) = \frac{\omega_0}{v_p} + \frac{\omega - \omega_0}{v_g} + \frac{\alpha}{2}(\omega - \omega_0)^2 \tag{2}$$

   where ω_0, v_p, v_g and α are constants. An electromagnetic field is excited inside this medium with a gaussian source such that the complex electric field at $x = 0$ is given by:

$$\mathbf{E}(x=0,t) = \hat{z}E_0\exp(i\omega_0 t)\exp(-t^2/\tau_0^2) \tag{3}$$

   The underlying principle of analyzing propagation of a non-monochromatic wave is to treat it as a superposition of monochromatic wave. This is easily accomplished by using some simple concepts from Fourier analysis.
   a) Express the electric field at $x = 0$ as a superposition of electric fields corresponding to monochromatic excitations, i.e. obtain an expression for $\tilde{\mathbf{E}}(\omega)$ in:

$$\mathbf{E}(x=0,t) = \int_{-\infty}^{\infty} \tilde{\mathbf{E}}(\omega)\exp(i\omega t)\mathrm{d}\omega \tag{4}$$

   b) Each of the monochromatic field inside the integral in Eq. 4 would propagate with a propagation constant given by Eq. 2. Consequently, the net electric field at $x > 0$ can be expressed as:

$$\mathbf{E}(x,t) = \int_{-\infty}^{\infty} \tilde{\mathbf{E}}(\omega)\exp(i\omega t - ik(\omega)x)\mathrm{d}\omega \tag{5}$$

   Using your answer to part (a) and Eq. 2, show that $\mathbf{E}(x,t)$ is given by:

$$\mathbf{E}(x,t) = \hat{z}E_0\frac{\exp(i\omega_0(t - x/v_p))}{(1 + 2i\alpha x/\tau_0^2)^{1/2}}\exp\left(-\frac{(t - x/v_g)^2}{\tau_0^2 + 2i\alpha x}\right) \tag{6}$$

c) Obtain an expression for the amplitude of the electric field as a function of time t at a fixed x. Show that the amplitude is a gaussian in time with a temporal width $\tau(x)$ larger than τ_0 (temporal width at $x = 0$). Quantify this broadening by computing $\Delta\tau(x)/\Delta x$ for large x.

Problems on reflection and transmission of EM waves

5) **Normal Incidence:** Consider an interface at $z = 0$ between two non-magnetic media as shown in Fig. 1. A plane electromagnetic wave with *real* electric field given by $\mathbf{E}_{\text{inc}} = \hat{x}E_0 \cos(\omega t - kz)$ is incident at the interface normally from $z < 0$:

a) Compute the incident *real* magnetic field and express k in terms of ω.

b) Compute the *real* transmitted and reflected electric fields.

6) **Incidence on a perfectly conducting slab:** An electromagnetic plane wave with electric field $\mathbf{E}_{\text{inc}} = E_0(8\hat{x} + 6\hat{y} + 5\hat{z})\sin(\omega t - \omega(3x - 4y)/5c)$ is incident on a perfectly conducting slab as shown in Fig. 2.

a) Compute the net *real* electric fields in region I, II and III.

b) If the conductor is *not* perfectly conducting, but has a finite conductivity $\sigma = 2\omega\epsilon_0$ and permittivity $\epsilon = \epsilon_0$, estimate the thickness of the slab so as to ensure that the electric field in region III is nearly 0 (assume $\exp(-5) \approx 0$).

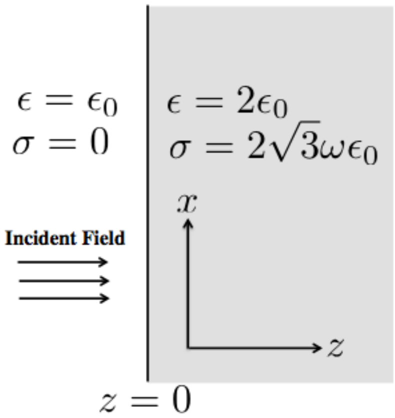

Figure 1: Normal Incidence

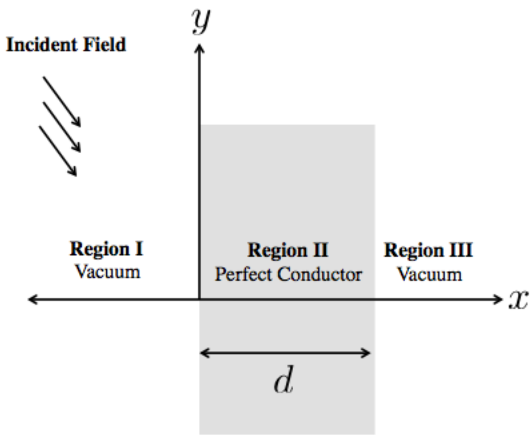

Figure 2: Incidence on a perfectly conducting slab