ELL212 - Tutorial 3 Solutions, Sem II 2015-16

1)

$$\nabla f = (y + xy^3)\hat{e}_x + \left(x + \frac{3}{2}x^2y^2\right)\hat{e}_y$$
$$f(1,1) - f(0,0) = 1 + \frac{1}{2} - 0 = \frac{3}{2}$$
For the path $y = x^n,\ \mathrm{d}y = nx^{n-1}\mathrm{d}x$ and $\mathrm{d}\vec{l} = (\hat{e}_x + nx^{n-1}\hat{e}_y)\mathrm{d}x$
$$\int_{P_1}^{P_2} \nabla f \cdot \mathrm{d}\vec{l} = \int_0^1 \left(x^n + x^{3n+1} + nx^{n-1}\left(x + \frac{3}{2}x^{2n+2}\right)\right)\mathrm{d}x$$
$$= \frac{1}{n+1} + \frac{1}{3n+2} + \frac{n}{n+1} + \frac{3}{2}\frac{n}{3n+2} = \frac{3}{2}$$

2)

$$\nabla \cdot \vec{v} = 4y$$
$$\int_V \nabla \cdot \vec{v}\mathrm{d}V = \int_0^3 \int_0^1 \int_0^{1-x} 4y\ \mathrm{d}y\mathrm{d}x\mathrm{d}z$$
$$= \int_0^3 \mathrm{d}z \int_0^1 2(1-x)^2\ \mathrm{d}x = 2$$
Let the surfaces be labelled as $1 : z = 0, 2 : z = 3, 3 : x = 0, 4 : y = 0, 5 : x + y = 1$

Let $\phi_i = \int_S \vec{v} \cdot \mathrm{d}\vec{S}$ over the i^{th} surface
$$\phi_1 = 0$$
$$\phi_2 = \int_0^1 \int_0^{1-x} 3y\ \mathrm{d}y = \frac{1}{2}$$
$$\phi_3 = 0$$
$$\phi_4 = 0$$
$$\phi_5 = \int_0^3 \int_0^1 [x(1-x) + (1-x)^2]\ \mathrm{d}x = \frac{3}{2}$$
$$\oint_S \vec{v} \cdot \mathrm{d}\vec{S} = \frac{3}{2} + \frac{1}{2} = 2$$

3) a)

For $r \neq 0,\ \nabla \cdot \left(\frac{1}{r^2}\hat{e}_r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = 0$

For $r = 0,$ Consider a small spherical volume of radius a and centre $(0,0,0)$
$$\int_V \nabla \cdot \left(\frac{1}{r^2}\hat{e}_r\right)\mathrm{d}V = \frac{1}{a^2}4\pi a^2 = 4\pi \implies \nabla \cdot \left(\frac{1}{r^2}\hat{e}_r\right) = 4\pi\delta^3(\vec{r})$$

b)

$$\nabla f = \frac{\partial f}{\partial r}\hat{e}_r = -\left(\frac{\lambda r + 1}{r^2}\right)\exp(-\lambda r)\hat{e}_r$$
$$\nabla \cdot \nabla f = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\lambda r + 1}{r^2}\exp(-\lambda r)\right) = \frac{\lambda^2}{r}\exp(-\lambda r) \text{ for } r \neq 0$$
For $r = 0,$ Consider a small spherical volume of radius a and centre $(0,0,0)$
$$\int_V \nabla^2 f\ \mathrm{d}V = -\frac{\lambda a + 1}{a^2}\exp(-\lambda a)4\pi a^2 \approx -4\pi$$
Hence $\nabla^2 f = \frac{\lambda^2}{r}\exp(-\lambda r) - 4\pi\delta^3(\vec{r})$

4)* Both are non-conservative

5)*

Use $\nabla \cdot (\Phi \nabla \Phi) = |\nabla \Phi|^2 + \Phi \nabla^2 \Phi = |\nabla \Phi|^2$

Thus $\displaystyle\int_V |\nabla \Phi|^2 \mathrm{d}V = \int_V \nabla \cdot (\Phi \nabla \Phi) \mathrm{d}V = \int_S \Phi \nabla \Phi \cdot \mathrm{d}\vec{S}$

But at the surface of V, $\Phi = 0$

$\implies \displaystyle\int_V |\nabla \Phi|^2 \mathrm{d}V = 0$

$\implies \nabla \Phi = \vec{0}$

$\implies \Phi = 0$ (Since $\nabla \Phi = 0$ implies it is a constant and $\Phi = 0$ on the surface of V)