ELL212 - Tutorial 3, Sem II 2015-16

1) Consider a scalar function $f(x,y)$ defined by
$$f(x,y) = xy + \frac{x^2y^3}{2}$$
Compute the gradient of f, ∇f. Verify that
$$\int_{P_1,C}^{P_2} \nabla f \cdot \mathrm{d}\vec{l} = f(P_2) - f(P_1)$$
where $P_1 = (0,0)$ and $P_2 = (1,1)$ and C is a contour joining P_1 and P_2 via the curve $y = x^n$.
2) Compute the divergence of the following vector function:
$$\vec{v} = xy\hat{x} + y^2\hat{y} + yz\hat{z}$$
Verify the Gauss' Divergence theorem
$$\int_V \nabla \cdot \vec{v}\ \mathrm{d}V = \oint_S \vec{v} \cdot \mathrm{d}\vec{S}$$
Over the volume of a prism as shown in Fig. 1

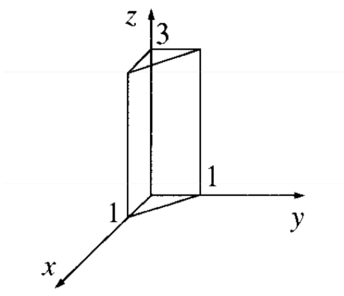

Fig. 1. Figure for **Q2**

3) a) Prove the following identities:
   i) In spherical coordinates ($\delta^3(\mathbf{r})$ is the 3D Dirac Delta function)
$$\nabla \cdot \left(\frac{1}{r^2}\hat{r}\right) = 4\pi\delta^3(\mathbf{r})$$
   ii) For any vector field $\vec{v}$ and scalar field ψ:
$$\nabla(\vec{v}\psi) = \nabla\psi \cdot \vec{v} + \psi\nabla \cdot \vec{v}$$
   b) For the scalar function
$$f(r,\theta,\phi) = \frac{\exp(-\lambda r)}{r}$$
   Compute ∇f and $\nabla^2 f$. [Hint: Use the result of part a]

4)* By calculating the curl of the following vector functions, classify them as conservative or non-conservative:
   a) $\vec{v}(r,\theta,\phi) = r^2\sin\theta\hat{r} + 4r^2\cos\theta\hat{\theta} + r^2\tan\theta\hat{\phi}$
   b) $\vec{v}(x,y,z) = x^2\hat{x} + xy\hat{y} + z^2\hat{z}$
5)* Let $\Phi(\vec{r})$ be a scalar function satisfying
$$\nabla^2\Phi = 0$$
in some volume V. Additionally it is known that $\Phi = 0$ at the surface S of the volume V. Prove that $\Phi = 0$ everywhere inside the volume V. [Hint: consider the integral $I = \int_V |\nabla\Phi|^2 \mathrm{d}V$ and try to prove that it is zero subject to the above conditions.]