Resource Allocation in Communication Networks When Users are Strategic

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THESIS CERTIFICATE

This is to certify that the thesis titled **Resource Allocation in Communication Networks When Users are Strategic**, submitted by **Anil Kumar Chorppath**, to the Indian Institute of Technology, Madras, for the award of the degree of **Master of Science**, is a bonafide record of the research work done by him under my supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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ABSTRACT

We consider resource allocation in the uplink of a cellular system where the base station has one or more channel resources it can allocate to several users. The base station requires channel quality and queue length information to achieve an efficient channel resource allocation. Unlike channel quality, queue length information resides privately with users and users are assumed to be strategic. In this work, an efficient uplink resource allocation is achieved while making users report true queue size information. For this, the problem is formulated as a mechanism design problem where users are the agents and base station the social planner. To make the users report true values, payments in terms of money are extracted from them. Since the payments cause loss in overall utility of users we attempt to minimize the payments in our mechanism.

We design two mechanisms to efficiently allocate the channel resource, which is divisible, to users while minimizing the loss in utility due to payments. The users have concave valuation functions parameterized by a scalar value and report only those scalar values. We design mechanisms in the Groves class, which are allocatively efficient, strategy proof but not budget balanced by assuming that valuation functions are known to the social planner. Mechanisms which are almost budget balanced are obtained without sacrificing the desirable property of individual rationality. The mechanisms proposed are characterized by linear rebate functions included in payments. First, the proposed worst-case optimal mechanism minimizes the worst ratio of budget surplus to efficient surplus. Next, an optimal-in-expectation mechanism that minimizes the ratio of expected budget surplus to expected efficient surplus is also proposed and compared with the worst-case optimal mechanism. Both the mechanisms are designed in a convex optimization framework. Numerical solutions for the coefficients of linear rebate function, worst-case efficiency loss and expected efficiency loss are obtained. In the special case of indivisible goods, the mechanisms fall back to those proposed by Moulin, by Guo & Conitzer and by Gujar & Narahari. Extension of the proposed mechanisms to the more realistic scenario where the valuation functions are private to agents is also analyzed. Issues in designing more competitive but inefficient allocation mechanisms and mechanisms without money as payment are also discussed.

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ABBREVIATIONS

BB	Budget Balance
DSIC	Dominant Strategy Incentive Compatible
AE	Allocative Efficiency
VP	Voluntary Participation
VCG	Vickrey-Clarke-Groves
d'AGVA	d'Aspremont-G'erard-VAret
SSVCG	Scalar Strategy Vickrey-Clarke-Groves
F	Feasibility
NEP	Nash Equilibrium Point
OIE	Optimal-in-expectation
WCO	Wost-case optimal

CHAPTER 1

Introduction

In many applications especially involving distributed systems, resource allocation depends on private information held by the end systems or users. There are several recent works which focussed on situations where these end users are strategic (see [1], [2] and [3]). Strategic users may misrepresent their private information so as to maximize their own utility even if at the expense of aggregate system utility [4]. In many practical applications like spectrum allocation, public good allocation, network resource allocation etc., the resource is shared by many users, i.e., it is divisible. An efficient allocation of a perfectly divisible resource to a number of strategic users is considered here.

1.1 Strategic behavior of users in uplink of a cellular system

We consider resource allocation in the uplink of a cellular system shown in Figure 1.1 which is a multiple access channel (MAC). In a cell, the base station has one channel resource it can allocate to the users. The channel resource can be frequency or time shared between the users, and thus, it can be considered as perfectly divisible. The resource allocation depends mainly on two parameters [5]. One parameter is the channel quality between the users and the base station. This information will be obtained by base station using the pilot symbols sent from the users. Another parameter is the number of packets waiting to be served in the queue of the users. This residual queue size is known

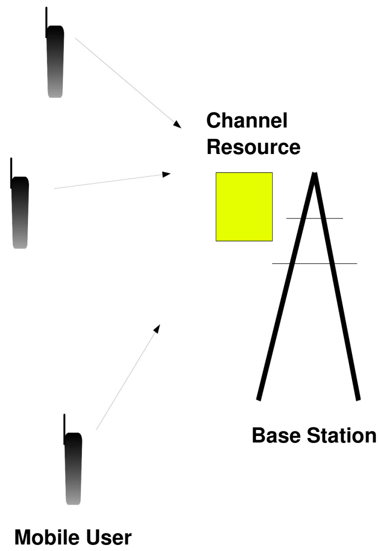

Figure 1.1: Uplink of a Cellular System

only to the individual users and should be reported by them to the base station [6]. A user having more residual packets in the queue, values the channel resource more. Thus, each user has an incentive to report higher queue sizes, so that it will get a higher share of the resource, to shed more packets. Here, the users are assumed to be strategic, and thus, interested only in their utility rather than overall system utility. In this scenario, the aim is to achieve efficient uplink channel resource allocation while making users report true queue size information. The theory of mechanism design is concerned with obtaining outcomes that meet a certain system level goal, despite the fact that individuals will pursue only their self-objectives. It contains a social planner who collects reported values from agents, knows (or assumes) value functions of agents, and allocates the available resources. Therefore, the problem above is equivalent to solving a mechanism design problem where users are the agents and base station the social planner.

Here, an efficient or socially optimal allocation of the resource is to be achieved. For this, mechanisms require the users to report true private values. To make the users report true values, payments in terms of money or the resource to be allocated itself, are extracted from them. In the quasi-linear environment considered here, the utility of an agent is the value obtained from allocation minus the payment. Hence, the payments causes loss in overall utility of users. Here, we assume that information about shape of value function of users, which is required for allocation and calculation of payments, is common knowledge. Therefore, the users need to report only a real value. In this work, an efficient allocation of divisible resource is done in a distributed system based on true scalar value reported by the end users while minimizing loss in overall utility due to payments. The solution obtained for a general scenario, will also solve the problem in the uplink of a cellular system, which motivated this work.

1.2 Mechanism design

Mechanism design has a setting where a social planner faces the problem of aggregating the reported values of strategic multiple agents into a collective system level outcome when the actual values are private to agents. It is a sub branch of game theory where, the rules of the induced game in the mechanism, are designed to achieve a socially desirable outcome. Thus, mechanism design can be viewed alternatively as reverse engineering of games. Recently, widespread interest is found in using mechanism design for modeling, analyzing and solving problems in network resource allocation and network economics which are decentralized in nature (see [1],[6] and [7]).

An allocatively efficient mechanism does efficient or socially optimal allocation of the resource. While doing this, it is to be ensured that the allocation is based on true private information reported by the agents. If in a mechanism, the agents are incentivised such that it is the dominant strategy of every agent to reveal true private information, then the mechanism is Dominant Strategy Incentive Compatible (DSIC) or strategy-proof. One way used in mechanism design to make the agents to reveal true private information is to incentivise them with payments. However, the payments causes loss in overall utility of agents. Therefore, in applications where there is no owner of the resource who will collect payments as revenue, and the users or systems involved wanted to efficiently allocate resource among themselves, the payments are undesirable. A mechanism is budget balanced (BB) if the net payment from agents to social planner or budget surplus is zero. Thus, in situations considered here, like the resource allocation problem in the previous section, ideally we would like to efficiently allocate the resource to users using true values reported by them while making the loss in utility due to payments zero.

1.3 Budget balancing problem

Groves mechanisms [8] are the only allocatively efficient and strategy-proof mechanisms in quasi linear environment. However, they are not budget balanced. In fact, the Vickrey-Clarke-Groves (VCG) mechanism (see [9] and [10]), a member of Groves class of mechanisms, maximizes the total payments from the agents to the social planner. The Green-Laffont impossibility theorem [11] says that there is no mechanism in a quasi-linear environment that is strategy-proof, achieves allocative efficiency, and is budget balanced.

An incentive compatible mechanism gives appropriate incentive to users for eliciting truth from them. These mechanisms require a commodity for which agents have extremely large demand known as '*numeraire commodity*' to incentivise them. Agents are charged in terms of this numeraire commodity for their consumption of original resource to be allocated, such that their overall utility is maximized when they report true information. In pricing schemes, a common way is to model money as a numeraire commodity

[7]. In this work, we impose payments in terms of money and try to minimize the loss in overall utility of agents caused by these payments. However, in these schemes, where money is used as a numeraire commodity, we require the prices to be updated at the rate of change in network topology. Price and Javidi [6] take another approach and use mechanisms in the context where the numeraire commodity is the downlink rates, for eliciting true uplink queue information in uplink of a cellular system. They model that there is higher demand of downlink rate over uplink in most communication systems, and come up with this approach. However, this approach causes a reduction in allocated downlink rates from the optimal point. In Section 6, we discuss issues related to implementing payment in terms of uplink channel resource itself.

1.4 Related work and review

In [5], optimal resource allocation is obtained in a communication system when complete channel and queue information are available. Recently, with the emergence of more and more distributed systems, resource allocation problems arise, where the users are strategic. Widespread interest is found in using game theory and mechanism design to solve these problems (see [2],[3] and [6]). In [6], efficient uplink rate allocation is achieved in the presence of strategic users with private uplink queue information. The payments are derived in terms of downlink rates.

A large literature in mechanism design has focussed on the impossibility result of Green & Laffont [11]. The d'Aspremont-Gérard-Varet (d'AGVA) [12] mechanism achieves allocatve efficiency and budget balance, but implements a Bayesian-Nash equilibrium. Kalagnanam [13] obtains budget balance by sacrificing allocative efficiency and DSIC properties in the context of generalized Vickrey auction where there are several buyers and sellers. Budget balanced but allocatively inefficient mechanisms where obtained by Faltings [14] for randomly generated social choice problems.

Guo & Conitzer [15] proposed almost budget balanced mechanisms for a setting where, there are m homogeneous indivisible goods to be allocated to n agents having unit demand, where $m < n$. A redistribution function was proposed which essentially redistribute back the VCG payments to agents. They maximized the worst-case (minimum) rebate redistribution fraction relative to the VCG payments and obtains an almost budget balanced mechanism. The worst case optimal mechanism was designed subject to the constraint that the resulting mechanism is feasible and agents are voluntary participating. The conditions on possible deterministic rebate functions for mechanisms in the Groves class, with properties anonymity and DSIC, were proposed by them. They used linear rebate function and proved the optimality of the linear rebate function among all possible rebate functions for Groves class of mechanism in the setting they considered.

Moulin [16] independent of Guo & Conitzer [15] proposed mechanisms within the Groves class for the same setting. A worst case optimal mechanism was proposed in the sense that it minimizes the worst (maximum) ratio of budget surplus (sum of payments) to efficient surplus (sum of valuations). The mechanism was designed subject to the same constraints and obtained the same linear rebate function as Guo & Conitzer [15]. Gujar & Narahari [18] proposed a mechanism for allocation of m heterogeneous indivisible goods to n agents again with unit demand where $m < n$. An agent reports only a scalar value and a scaling based correlated valuation vector of size $1 \times m$ is created from it. They also proposed a linear redistribution mechanism for this setting and proved it is worst case optimal among all the Groves redistribution mechanisms which are feasible, voluntary participating and anonymous.

Guo & Conitzer [17] proposed a different redistribution mechanism that maximizes the *average* rebate redistributions. Once again the redistribution for an agent was linear in the reported values of all other agents.

We extend the linear redistribution mechanisms proposed by Moulin [16] and Guo & Conitzer [15] to the case when the resource is perfectly divisible. The valuation function of an agent, which is assumed to be known to the social planner, is any concave function parameterized by a scalar value. The agents report only the scalar values. Our generalized mechanisms reduce to those proposed by Moulin, Guo & Conitzer, and Gujar & Narahari in the corresponding special settings.

The assumption that the valuation function is known to the central planner is often unrealistic in practical distributed systems like the Internet. Reporting the entire valuation function, which is infinite dimensional, is a considerable communication burden to the system (see Johari & Tsitsiklis [19]). If the allocation mechanism is based only on reported real values in quasi-linear environment, then dominant strategy implementation is not possible and the central planner should rely on Nash equilibria played by agents. Sanghavi & Hajek [20] focused on one-dimensional real-valued bids as payment by agents, and studied the Nash equilibrium implementation. Kelly, Maulloo & Tan [21] proposed a mechanism where the central planner creates *surrogates* for the valuation function from the one-dimensional bids. The allocation and payment are derived using these surrogate valuation functions. They consider a network resource allocation problem where agents are price taking, i.e., agents sees a fixed price set by network. Johari & Tsitsiklis [4] consider strategic agents, where agents are aware of how price is influenced by their bid, in the Kelly setting and established a worst case efficiency at Nash equilibrium. Yang & Hajek [22] proposed a VCG-Kelly mechanism by combining the one-dimensional bid idea of Kelly et al. with the VCG mechanism for the network rate allocation problem.

They achieve socially optimal allocation at a unique Nash Equilibrium Point (NEP). NEP is shown to be globally stable through a dynamic algorithm. Johari & Tsitsiklis [19] analyzed the more general convex environment, proposed a scalar strategy VCG (SSVCG) mechanism. Necessary and sufficient conditions for the existence of an efficient Nash equilibrium were obtained. Under further assumptions on the constraint set, they established that all NEPs are socially optimal.

Guo & Conitzer in a subsequent work in [23] proposed mechanisms outside the Groves class. They showed that better competitive ratios can be achieved by allowing inefficient allocation. One strongly budget balanced method with an inefficient allocation partitions goods and agents into two groups, allocates group-wise, and redistributes payments from one group as rebates to the other group. Another such method burns a part of the goods.

1.5 Contributions of the thesis

1. We formulated the problem of efficient resource allocation in uplink of a cellular system in the presence of strategic users as a mechanism design problem. To solve this problem we proposed mechanisms in the Groves class, which are AE, DSIC and are almost budget balanced, for allocation of a single divisible resource to a number of agents. The valuation function of all the agents are taken to be a general concave function which is parameterized by the scalar values reported by them. A convex optimization framework is used to design all the proposed mechanisms.

2. Two mechanisms are proposed in the divisible good setting, one is worst case optimal and the other one is optimal-in-expectation. The proposed mechanisms are essentially redistribution mechanisms where a rebate function in payments redis-

tributes the VCG payment back to agents. Both the proposed mechanisms use linear redistribution functions.

(a) The proposed worst-case optimal mechanism minimizes the worst ratio of budget surplus to efficient surplus for allocation of a perfectly divisible resource.

(b) The proposed Optimal-in-expectation mechanism minimizes the ratio of expected budget surplus to expected efficient surplus for divisible resource allocation.

In the special setting of allocation of several indivisible homogeneous goods to multiple agents with unit demand, the mechanisms proposed here falls back to optimal mechanisms proposed by Moulin [16] and Guo & Conitzer [15]. Gujar & Narahari [18] proposed mechanism for allocation of m heterogeneous goods to n agents when the valuations of an agent to the goods have scaling based correlation. With an additional constraint on possible allocations their mechanism fits to our framework in the special setting.

3. In many practical applications, the assumption that the valuation function is known to the central planner does not hold. The mechanisms proposed by us can be extended to this setting by using the surrogate valuation function constructed from the reported scalar values and an almost budget balanced and efficient Nash implementation can be obtained.

CHAPTER 2

Mechanism Design for Resource Allocation

In resource allocation perspective, mechanism design is used as a mathematical tool to model, analyze and solve decentralized problems. The mechanism design setting for resource allocation in a quasi-linear environment is described in the following section.

2.1 The setting

There is a social planner and n agents, $\{1, 2, \ldots, n\} = N$, carrying private information or preferences $\{\theta_1, \theta_2, \ldots, \theta_n\}$ about a resource, with each $\theta_i \in \Theta_i$, a compact subset of $\mathbb{R}_+$. Agents make a collective choice from O which is *outcome set*. Agent i receives an allocation of resource a_i that will depend on the entire preference profile $\underline{\theta}$ of size $n \times 1$. Let $\underline{a}$ be the allocation vector of size $n \times 1$ and let A be the set of all possible allocations. Let $\underline{\theta}_{-i} \in \Theta_{-i}$ denote the preference vector with zero in the i^{th} position of $\underline{\theta}$. An allocation vector obtained by considering $\underline{\theta}_{-i}$ is denoted $\underline{a}_{-i} \in A_{-i}$. An agent obtains a valuation $v_i(a_i; \theta_i)$, depending on her bid and the allocation received. Here, we assume that the shape of valuation function is known to the social planner. So each agent reports only her preference. An agent makes a payment p_i to the social planner depending on entire preference profile $\underline{\theta}$. Payment from agents towards the social planner are taken to be positive. The outcome set $O = \{\underline{a}, \underline{p}\}$, where $\underline{p}$ is the payment vector of all the agents. Agents have utility u_i which denotes the payoff the agents derive from the allocation given their preferences, and considering payments. In a quasi-linear environment, the valuation and payments in the utility expression of an agent are related linearly, i.e,

$$u_i(\underline{\theta}) = v_i(a_i; \theta_i) - p_i(\underline{\theta}).$$

There is a common prior distribution of agent's preferences $\Phi \in \Delta\Theta$, where $\Theta = \Theta_1 \times \Theta_2 \times .... \times \Theta_n$. An individual belief function is derived from Φ for each agent, given her preference. The key assumptions in mechanism design are rationality and intelligence of participating agents. The rationality assumption makes an agent to strategise only for its own objectives. An agent is intelligent if she can make inference about the game induced in the mechanism that a game theorist can make by knowing everything a game theorist knows. $\Phi, \Theta_1, \Theta_2 \ldots \Theta_N$ and $u_1(.), u_2(.) \ldots u_N(.)$ are assumed to be common knowledge.

2.2 Properties of mechanisms

Some properties of mechanisms in quasi-linear environment are the following.

1. Allocative Efficiency (AE) - An efficient (or) optimal allocation $a^*(\theta)$ maximizes total value of all the agents.

   $$a^*(\underline{\theta}) = \arg\max_{a \in A} \sum_{i \in N} v_i(a_i, \theta_i).$$

   A mechanism is allocatively efficient if the allocation of the resource to agents is efficient.

2. Dominant Strategy Incentive Compatible (DSIC) - The mechanism is DSIC if it incentivise agents such that truth revelation becomes dominant strategy of all the agents.

3. Budget Balance (BB)- If the net payments from agents towards the social planner is zero then the mechanism is said to be BB i.e.

$$\sum_{i \in N} p_i(\underline{\theta}) = 0.$$

If there is a positive net payment from agents to the social planner (budget surplus) then the mechanism is said to be Weak Budget Balanced (WBB) or Feasible (F). This means that no external subsidy of money or resource is required to sustain the mechanism.

4. Individual Rationality (or) Voluntary Participation (VP): This property ensures that the utility of all agents should be greater than or equal to the utility they would get by dropping out of the mechanism. The utility that agents get by not participating in the mechanism is usually taken to be zero. Thus for VP,

$$u_i(\underline{\theta}) \geq 0, \ \forall i \in N, \ \forall \ \underline{\theta}. \tag{2.1}$$

2.3 Groves mechanisms

In quasi-linear environment, the payment of each agent in Groves mechanisms [8] is such that the resulting mechanism becomes allocatively efficient and DSIC. The payment of i^{th} agent is given by

$$p_i(\underline{\theta}) = h_i(\underline{\theta}_{-i}) - \sum_{j \neq i} v_j(a_j^*(\underline{\theta}), \theta_j)$$

where $h_i : \Theta_{-i} \to \mathbb{R}$ is any arbitrary function. Note that, depending on values given by $h_i(\theta_{-i})$ the budget balance of the mechanism changes.

In the celebrated Vickrey-Clarke-Groves (VCG) mechanism which is a member of Groves class(see [9] and [10]), $h_i(\underline{\theta}_{-i})$ is set as

$$h_i(\underline{\theta}_{-i}) = \sum_{j \neq i} v_j(a^*_{-i,j}(\underline{\theta}_{-i}), \theta_j)$$

where

$$\underline{a}^*_{-i}(\underline{\theta}_{-i}) = \arg \max_{\underline{a}_{-i} \in A_{-i}} \sum_{j \neq i} v_j(a_{-i,j}, \theta_j),$$

and $a_{-i,j}$ is the jth component of $\underline{a}_{-i}$. $\underline{a}^*_{-i}(\underline{\theta}_{-i})$ is the efficient allocation when i^{th} agent is not considered for allocation. The VCG payment for an agent is the difference the agent makes to the aggregate value of other agents by participating in the mechanism.

2.4 Redistribution mechanisms for divisible resource allocation

Consider a perfectly divisible good to be allocated to agents $\{1, 2, \ldots, n\} = N$. Since we consider mechanisms only within the class of Groves mechanisms, which are DSIC, we may assume that all agents report their true preferences.[1] Thus, agents reports scalar bids $\{\theta_1, \theta_2, \ldots, \theta_n\}$ with each $\theta_i \in \Theta_i$. The valuation functions considered is assumed to satisfy following assumption.

Assumption 1. *The valuation function $v_i(\cdot, \theta_i)$ that maps $a_i \mapsto v_i(a_i; \theta_i)$ is concave, nondecreasing in $[0, \infty)$, differentiable in $(0, \infty)$, and satisfies $v_i(a_i, 0) = 0$.*

Let V be set of functions that satisfy Assumption 1. We focus our attention on mechanisms that are allocative efficient, DSIC, and budget balanced to the extend possible.

[1] In Section 6 we discuss mechanisms outside this class.

Groves class of mechanisms are the only efficient allocation mechanisms that are DSIC in quasi-linear environment. To obtain a mechanism which is more budget balanced in Groves class, a rebate (or) redistribution was introduced by Moulin in [16] in the payment of VCG mechanism. A rebate function determines the redistributions back to the agents of a portion of the VCG payments. The choice of these rebates should be such that the DSIC property of the mechanism is preserved. Moreover, the mechanism should be anonymous, i.e., two agents with identical bids should get identical rebates. The condition for obtaining an anonymous and DSIC rebate function is given in the following theorem.

Theorem 1. *Suppose that agents bid scalar values and that the scalar parameterized value functions satisfy Assumption 1. Then, any mechanism with deterministic and anonymous redistributions is DSIC if and only if the rebate function can be written as*

$$r_i = f(\theta_1, \theta_2, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n)$$

for some f with arguments satisfying $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_{i-1} \geq \theta_{i+1} \geq \ldots \geq \theta_n$.

Proof. The proof is identical to that in Guo & Conitzer [15]. □

The payment for the new mechanism with rebates, one that remains within the Groves class of mechanisms, is given by

$$p_i(\underline{\theta}) = \sum_{j \neq i} v_j(a^*_{-i,j}(\underline{\theta}_{-i}), \theta_j) - \sum_{j \neq i} v_j(a^*_j(\underline{\theta}), \theta_j) - r_i(\underline{\theta}_{-i}). \qquad (2.2)$$

The rebate function in Theorem 1 should preserve all the desirable properties of the VCG mechanism. These are the following.

1) Feasibility (F) : As given in previous section this property ensures that there is a net payment (budget surplus) from the agents to the mechanism:

$$\sum_{i \in N} p_i(\underline{\theta}) \geq 0, \ \forall \ \underline{\theta}. \tag{2.3}$$

Substitution of equation (2.2) in equation (2.3) yields

$$\sum_{i \in N} \sum_{j \neq i} v_j(a^*_{-i,j}(\underline{\theta}_{-i}), \theta_j) - (n-1) \sum_{i \in N} v_i(a^*_i(\underline{\theta}), \theta_i) - \sum_{i \in N} r_i(\underline{\theta}_{-i}) \geq 0, \ \forall \ \underline{\theta},$$

or, equivalently,

$$\begin{aligned} \sum_{i \in N} r_i(\underline{\theta}_{-i}) \ &\leq \ \sum_{i \in N} \sum_{j \neq i} v_j(a^*_{-i,j}(\underline{\theta}_{-i}), \theta_j) - (n-1) \sum_{i \in N} v_i(a^*_i(\underline{\theta}), \theta_i) \\ &=: \ p_{VCG}(\underline{\theta}), \ \forall \ \underline{\theta}, \end{aligned} \tag{2.4}$$

where $p_{VCG}(\underline{\theta})$ is the total VCG payment by all the agents.

2) Voluntary Participation (VP): From 2.1,

$$u_i(\underline{\theta}) = v_i(a^*_i(\underline{\theta}), \theta_i) - p_i(\underline{\theta}) \geq 0, \ \forall i \in N, \ \forall \ \underline{\theta}. \tag{2.5}$$

Substitution of equation (2.2) in equation (2.5) yields

$$\sum_{j \in N} v_j(a^*_j(\underline{\theta}), \theta_j) - \sum_{j \neq i} v_j(a^*_{-i,j}(\underline{\theta}_{-i}), \theta_j) + \ r_i(\underline{\theta}_{-i}) \geq 0, \ \forall i \in N, \ \forall \ \underline{\theta},$$

or, equivalently,

$$\begin{aligned} r_i(\underline{\theta}_{-i}) \quad &\geq \quad \sum_{j \neq i} v_j(a^*_{-i,j}(\underline{\theta}_{-i}), \theta_j) - \sum_{j \in N} v_j(a^*_j(\underline{\theta}), \theta_j) \\ &=: \quad n_i(\underline{\theta}), \ \forall \ i \in N, \ \forall \ \underline{\theta}. \end{aligned} \tag{2.6}$$

Adding all the n constraints in equation (2.6) and using equation (2.4), we get

$$p_{VCG}(\underline{\theta}) - \sum_{i \in N} v_i(a^*_i(\underline{\theta}), \theta_i) \leq \sum_{i \in N} r_i(\underline{\theta}_{-i}) \leq p_{VCG}(\underline{\theta}), \ \forall \ \underline{\theta}.$$

We shall consider the case of a single divisible good allocated to a number of agents. The only assumption is that the valuation function satisfies Assumption 1. The Moulin [16] and Guo & Conitzer [15] mechanisms are for allocation of m homogeneous indivisible goods to n agents, each demanding a unit good, where $m \leq n$. This fits within our framework if we divide the single good into m equal parts with $m \leq n$ and take the piece-wise linear valuation function $v_i(a_i, \theta_i) = \theta_i \min\{a_i, 1/m\}$, i.e., each agent's valuation increases linearly, but saturates after reaching the threshold $1/m$. The Gujar & Narahari [18] mechanism is for allocation of m heterogeneous indivisible goods to n agents where valuations of an agent to goods have scalar based correlation. This mechanism also fits into our framework when we divide the good into m unequal parts, take the valuation function to be $v_i(a_i, \theta_i) = \theta_i a_i$, and impose the allocation constraint that each agent gets at most one of the unequal parts. Thus, our proposed generalizations in the next section fall back to those of Moulin [16], Guo & Conitzer [15], and Gujar & Narahari [18], in the appropriate special settings.

CHAPTER 3

Almost Budget Balanced Linear Redistribution Mechanisms

The redistribution function can take any form as specified in Theorem 1. A linear form of redistribution function was proposed by Moulin [16] and by Guo & Conitzer [15]. The latter authors showed that for the worst-case problem, linear redistribution mechanism is optimal among all Groves mechanisms that are feasible and individually rational. We too shall focus on a linear redistribution function. The rebate for the i^{th} agent is given by

$$r_i(\underline{\theta}_{-i}) = c_0 + c_1\theta_1 + \ldots + c_{i-1}\theta_{i-1} + c_i\theta_{i+1} + \ldots + c_{n-1}\theta_n$$

where $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n$. Consequently, we have

$$\sum_{i \in N} r_i(\underline{\theta}_{-i}) = nc_0 + \sum_{i=1}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i). \tag{3.1}$$

After substitution of equation (3.1) in equations (2.4) and (2.6), constraints F and VP in the linear redistribution case become

$$\begin{array}{ll}
\text{(F)} & nc_0 + \displaystyle\sum_{i=1}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i) \leq p_{VCG}(\underline{\theta}), \ \forall\ \underline{\theta}, \\
\text{(VP)} & c_0 + \displaystyle\sum_{j=1}^{i-1} c_j\theta_j + \sum_{j=i}^{n-1} c_j\theta_{j+1} \ \geq \ n_i(\underline{\theta}), \qquad \forall\ \underline{\theta},\ \forall i \in N.
\end{array}$$

Let $\underline{e}_k = (1, 1, \ldots, 1, 0, 0, \ldots, 0)$ with k 1s. Setting $\underline{\theta} = \underline{e}_0$, we get $c_0 = 0$ from F and VP constraints. Setting $\underline{\theta} = \underline{e}_1$, we get $p_{VCG}(\underline{\theta}) = 0$ and $n_i(\underline{\theta}) = 0$ for any $i \geq 2$. Therefore, using constraint F, we get $(n-1)c_1 \leq 0$. On the other hand, using constraint VP, we get $r_2(\underline{\theta}_{-2}) = c_1 \geq 0$, yielding $c_1 = 0$. Furthermore,

Lemma 1. *The following system of inequalities are equivalent.*

$$\begin{array}{lll} (a) & r_i(\underline{\theta}_{-i}) \geq n_i(\underline{\theta}), \quad \forall\ \underline{\theta},\ \forall\ i \in N. & \\ (b) & \displaystyle\sum_{i=2}^{k} c_i \geq 0, & k = 2, 3, \ldots, n-1. \end{array}$$

Proof. (a) $\Rightarrow$ (b): The definition of $n_i(\theta)$ in the right-hand side of equation (2.6) yields

$$n_i(\underline{\theta}) = \sum_{j \in N - \{i\}} v_j(a^*_{-i,j}(\underline{\theta}_{-i}), \theta_j) - \sum_{j=1}^{n} v_j(a^*_j(\underline{\theta}), \theta_j) \leq 0, \tag{3.2}$$

because $\underline{a}^*_{-i}(\underline{\theta}_{-i})$ is an inefficient allocation in comparison to $\underline{a}^*(\underline{\theta})$ when all the n agents are active.

Consider $\underline{\theta} = \underline{e}_k$ for $k = 2, 3, \ldots, n-1$. The rebates for these bids, i.e., after substitution in (3.1), are

$$r_{k+1}(\underline{\theta}_{-(k+1)}) = \sum_{i=2}^{k} c_i.$$

Moreover,

$$\begin{aligned} n_{k+1}(\underline{\theta}) &= \sum_{j \in N - \{k+1\}} v_j(a^*_{-(k+1),j}(\underline{\theta}_{-(k+1)}), \theta_j) - \sum_{j=1}^{n} v_j(a^*_j(\underline{\theta}), \theta_j) \\ &= \sum_{j=1}^{k} v_j(a^*_{-(k+1),j}(\underline{\theta}_{-(k+1)}), \theta_j) - \sum_{j=1}^{k} v_j(a^*_j(\underline{\theta}), \theta_j) \\ &= 0, \end{aligned}$$

because $v_j(a_j, 0) = 0$ for $j \geq k+1$, and therefore

$$\underline{a}^*_{-(k+1)}(\underline{\theta}_{-(k+1)}) = \underline{a}^*(\underline{\theta})$$

as a consequence of the fact that $\underline{\theta}_{-(k+1)} = \underline{\theta} = \underline{e}_k$. Substitution of these in the VP constraint yields $\sum_{i=2}^{k} c_i \geq 0$ for $k = 2, 3, \ldots, n-1$.

(b) $\Rightarrow$ (a): From Lemma 1 proved by Guo & Conitzer [15], if $\sum_{i=2}^{k} c_i \geq 0$ for all $k = 2, 3, \ldots, n-1$ then

$$c_2\theta_2 + \ldots + c_{i-1}\theta_{i-1} + c_i\theta_{i+1} + \ldots + c_{n-1}\theta_n \geq 0$$

for all $\theta_1 \geq \theta_2 \geq \theta_3 \geq \ldots \geq \theta_n$. Consequently, $r_i(\underline{\theta}_{-i}) \geq 0$ for all $i \in N$ and the reverse implication follows from equation (3.2). This proves the lemma. □

3.1 Worst-case optimal (WCO) mechanism

Moulin [16] proposed a mechanism that minimizes the *worst-case efficiency loss*. We shall now describe this objective. Let the efficient surplus be

$$\sigma_v(\underline{\theta}) = \sum_{i \in N} v_i(\underline{a}^*(\underline{\theta}), \theta_i).$$

The worst-case efficiency loss is the maximum ratio of budget surplus to the efficient surplus over all possible $\underline{\theta}$, i.e.,

$$L(n) = \max_{\underline{\theta}} \frac{\sum_i p_i(\underline{\theta})}{\sigma_v(\underline{\theta})}. \tag{3.3}$$

Moulin [16] minimized this objective function $L(n)$ subject to F and VP constraints, but under the homogeneous goods setting. We shall now generalize this to the perfectly divisible case with the linear redistribution constraint, i.e., we will solve

$$\min_{c_2,\ldots,c_{n-1}} \max_{\underline{\theta}} \frac{p_{VCG}(\underline{\theta}) - \sum_{i=2}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i)}{\sigma_v(\underline{\theta})} \tag{3.4}$$

subject to

1. $\sum_{i=2}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i) \leq p_{VCG}(\underline{\theta}),\ \forall\ \theta,$

2. $\sum_{i=2}^{k} c_i \geq 0,\ \forall\ k = 2, 3, \ldots, n-1.$

This min-max problem can be rewritten as a minimization problem by adding an additional constraint:

$$\min_{c_2,\ldots,c_{n-1},L(n)} \qquad L(n) \tag{3.5}$$

subject to

1. $\sum_{i=2}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i) \leq p_{VCG}(\underline{\theta}),\ \forall\ \theta,$

2. $\sum_{i=2}^{k} c_i \geq 0,\ \forall\ k = 2, 3, \ldots, n-1,$

3. $\sum_{i=2}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i) + L(n)\sigma_v(\underline{\theta}) \geq p_{VCG}(\underline{\theta}),\ \forall\ \underline{\theta}.$

In constraint 1) of problem (3.5), let $C_1(\underline{\theta})$ be a set of feasible coefficients for a given value of $\underline{\theta}$. This defines a half plane, a convex set. Thus the intersection of these half plane constraints $C_1 = \bigcap_{\underline{\theta}} C_1(\underline{\theta})$ is also a convex set. In constraint 3), if $C_2(\underline{\theta})$ is the set of feasible coefficients for a given $\underline{\theta}$, then $C_2 = \bigcap_{\underline{\theta}} C_2(\underline{\theta})$ is also a convex set, and $C_1 \bigcap C_2$, the set of coefficients that satisfy both constraints 1) and 3), is a convex set. Finally, the $n-2$ conditions in constraint 2) define a polygon, another convex set, and the

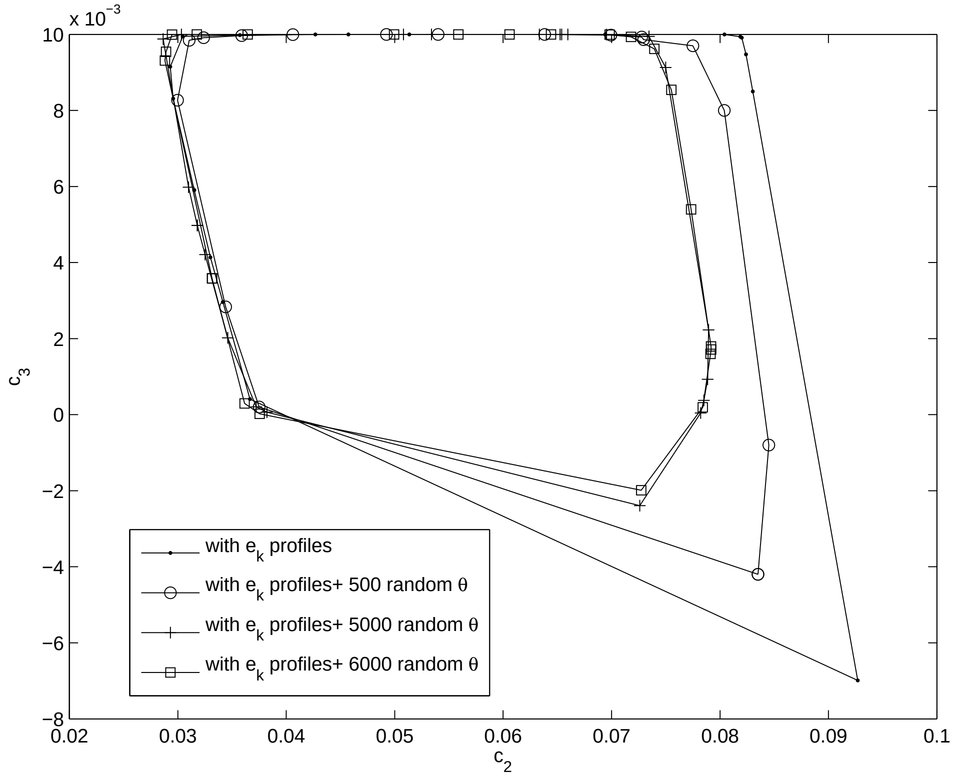

Figure 3.1: Feasible region of c_2 and c_3 for number of agents=8 obtained with different number of uniformly random generated $\underline{\theta}$'s and e_k profiles

minimization problem in (3.5) subject to constraints 1), 2) and 3) is a convex optimization problem. Let us denote the convex constraint set by C.

In problem (3.5), constraints 1) and 3) are a function of $\underline{\theta} \in \Theta^N$. What we then have is a continuum of half-space constraints whose intersections, along with those of constraint 2), yields the overall convex constraint set C. Guo & Conitzer [15] proved that the constraints obtained with $\underline{\theta}$ profiles $\underline{e}_k = (1, 1, \ldots, 1, 0, \ldots, 0)$ having k 1s, for $k = 0, 1, \ldots, n$, are enough to specify the feasible region in the case of indivisible goods. We too start with these $\underline{\theta}$ profiles. However, they do not fully characterize the feasible region for the divisible goods case. (See Figure 3.1 above). Additional constraints were obtained by sampling random values of $\underline{\theta}$ uniformly on Θ^N. This yields an approximation $\hat{C}$ of C. The convex optimization problem is then solved numerically subject to the approximate constraint that the coefficients lie in $\hat{C}$. More details can be found in section 5.

The natural question that arises is the goodness of the approximation $\hat{C}$ as the number of random samples increases. In Figure 3.1, the number of agents $n = 8$, the variables are $c_2, c_3, \ldots, c_7$ and $L(n)$, and for pictorial depiction, only the c_2-c_3 region is plotted after disregarding the constraint on other variables. Figure 3.1 gives a sequence of approximations to the feasible region for c_2 and c_3. The coarsest is the one that merely uses the $\underline{e}_k$ profiles. This region is progressively refined with 500, 5000, and 6000 samples of $\underline{\theta} \in [0,1]^n$. We observe that there is little difference between the regions for 5000 and 6000 samples.

3.1.1 Mean-field approximation and concentration

Constraints 1) and 3) are those that depend on $\underline{\theta}$. We now show that the nonlinear terms $p_{VCG}(\underline{\theta})$ and $\sigma_v(\underline{\theta})$ concentrate to constants for large n, under uniform sampling, and also identify a suitable mean-field approximation for intermediate values of n. Under the large n limit, or under the mean-field approximation, the constraints are linear in θ_i, and the convex optimization problem in (3.5) has a relaxation that is a simpler linear programming problem.

We begin with some motivating simulation results. Figures 3.2 and 3.3 provide the empirical histogram of $\sigma_v(\underline{\theta})$ and $p_{VCG}(\underline{\theta})$ as $\underline{\theta}$ is sampled uniformly from $[0,1]^N$ where we took $\Theta = [0,1]$, $v_i(\theta_i, a_i) = \theta_i \log(1 + a_i)$, with the constraint $\sum_{i \in N} a_i = 1$. The histograms are approximations of the probability density function (pdf) of the random variables $\sigma_v(\underline{\theta})$ and $p_{VCG}(\underline{\theta})$, respectively. The plots show pdfs for $n = 5, 20, 50, 200, 500$. A good mean-field approximation should approximate $\sigma_v(\underline{\theta})$ to a constant given by the location of the peak; similarly for $p_{VCG}(\underline{\theta})$. Further observe that as $n \to \infty$, the pdfs indicate that both random variables converge to the constant 1. We now substantiate these observations.

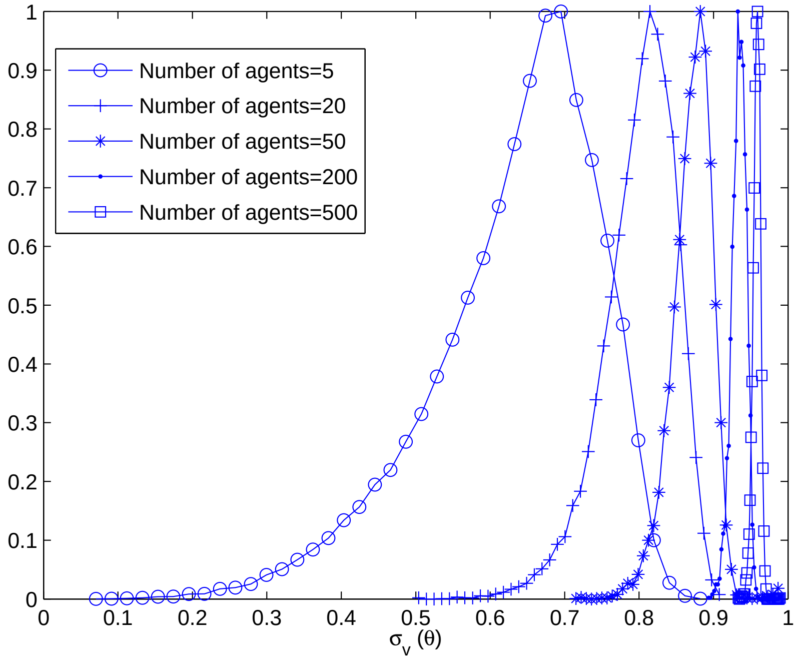

Figure 3.2: Histogram of $\sigma_v(\underline{\theta}\ )$ for different number of agents

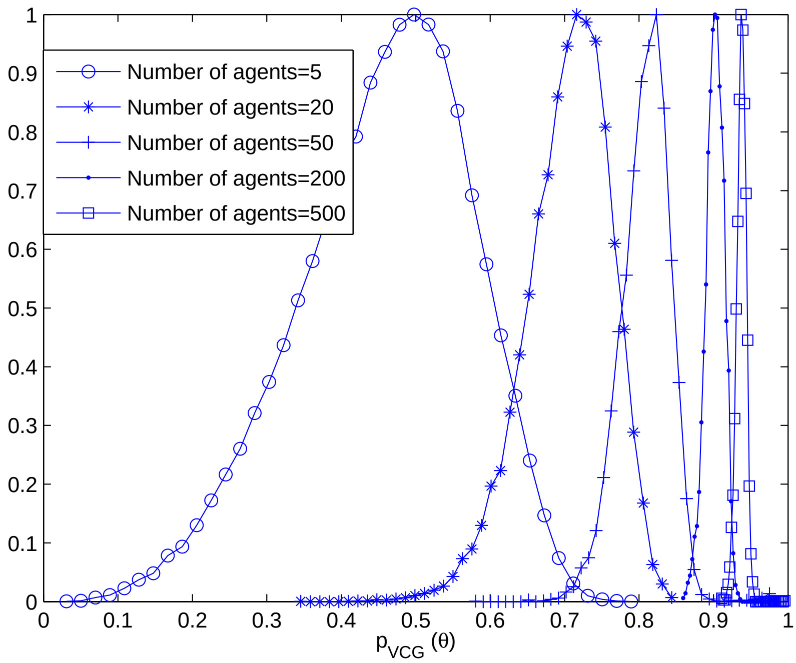

Figure 3.3: Histogram of $p_{VCG}(\underline{\theta})$ for different number of agents

Claim 1. *Let θ_i be independent and identically uniformly distributed on $\Theta = [0, 1]$. Let*

$v_i(\theta_i, a_i) = \theta_i \log(1 + a_i)$, *with the constraint that* $\sum_i a_i = 1$. *Then, for all* n, *the mean-field approximation for* σ_v *is*

$$\sigma_v \simeq \frac{n}{4}\left[-2\log\lambda(n) - 1 + \lambda^2(n)\right].$$

Moreover, $\lim_{n\to\infty} \sigma_v(\underline{\theta}) = 1$ *almost surely.*

Proof. The cumulative distribution function (cdf) for the iid θ_i is $F(x) = x,\ x \in [0,1]$. Given a realization $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$, denote the empirical cdf by $F(x; \underline{\theta})$. The mean-field approximation simply sets $F(x; \underline{\theta})$ to its expected value $F(x)$ for all n and x. The approximation is accurate as $n \to \infty$ as ascertained by the Glivenko-Cantelli theorem (see pages 85-108 in [24]):

$$\lim_{n\to\infty} \sup_{x\in\Theta} |F(x; \underline{\theta}) - F(x)| = 0 \text{ (almost surely).}$$

For a given realization $\underline{\theta}$, the solution to

$$\max \sum_i \theta_i \log(1 + a_i)$$

subject to the constraints $\sum_i a_i = 1$ and $a_i \geq 0$ is given by the optimal allocation

$$a_i(\underline{\theta}) = \left[\frac{\theta_i}{\lambda(n, \underline{\theta})} - 1\right]_+,$$

where $[x]_+ = \max\{x, 0\}$, and $\lambda(n, \underline{\theta})$ is chosen so that

$$\sum_{i=1}^{n} \left[\frac{\theta_i}{\lambda(n, \underline{\theta})} - 1\right]_+ = 1. \qquad (3.6)$$

When $\underline{\theta}$ is sorted in ascending order, the number of agents that bid values in the interval $(x, x+dx)$ is, under the mean-field approximation, given by $n\ dx$. Under this approximation, $\lambda(n, \underline{\theta}) = \lambda(n)$, i.e., it does not depend on the realization. The constraint in (3.6) then becomes

$$\int_0^1 n\ dx\ \left[\frac{x}{\lambda(n)} - 1\right]_+ = \int_{\lambda(n)}^1 n\ dx\ \left[\frac{x}{\lambda(n)} - 1\right] = 1,$$

which yields the quadratic equation $n(1-\lambda(n))^2 = 2\lambda(n)$. The only meaningful solution to this equation is the one in the interval $[0,1]$, which is $\lambda(n) = 1 + \frac{1-\sqrt{2n+1}}{n}$.

Next,

$$\begin{aligned}
\sigma_v(\underline{\theta}) &= \sum_{i=1}^{n} \theta_i \log(1 + a_i(\underline{\theta})) \\
&\simeq \sum_x n\ dx\ x \log\left(1 + \left[\frac{x}{\lambda(n)} - 1\right]_+\right) \\
&= n \int_{\lambda(n)}^1 x\ dx \log\left(\frac{x}{\lambda(n)}\right) \\
&= \frac{n}{4}\left[-2\log\lambda(n) - 1 + \lambda^2(n)\right],
\end{aligned}$$

where the summand in the approximate sum is obtained by summing over agents i whose θ_i fall in the interval $(x, x+dx)$, which we call bin with index x. The outer summation is over bin indices x. This approximation is almost surely correct as $n \to \infty$ because the integral is that of a bounded and continuous function. The last equality follows from the evaluation of the definite integral. After substitution of $\lambda(n) = 1 + \frac{1-\sqrt{2n+1}}{n}$ in the last expression and after taking limits, we obtain that $\sigma_v \to 1$ almost surely. This concludes the proof. □

In Table 3.1, we compare the mean-field approximation of $\sigma_v(\theta)$ with the peaks in

n	σ_v	Peak in Figure 3.2
20	0.8126	0.8185
50	0.8763	0.8822
200	0.9358	0.9326
500	0.9588	0.9599

Table 3.1: Comparing $\sigma_v(\underline{\theta})$ peak occurring points in histograms to the values obtained from Claim 1

Figure 3.3, for different number of agents. We observe that the mean-field approximation is quite accurate. We next consider $p_{VCG}(\underline{\theta})$.

Claim 2. *Under the assumptions in Claim 1, the mean-field approximation for p_{VCG} is*

$$\begin{aligned} p_{VCG} &\simeq n(n-1)/4\left[-2\log\lambda(n-1)+\lambda^2(n-1)\right. \\ &\qquad \left.+2\log\lambda(n)-\lambda^2(n)\right], \end{aligned}$$

and $\lim_{n\to\infty} p_{VCG}(\underline{\theta}) = 1$ *almost surely.*

Proof. We first note that

$$\begin{aligned} p_{VCG}(\underline{\theta}) &= \sum_{i\in N}\sum_{j\neq i}\theta_j\log(1+a_{-i,j}(\underline{\theta}_{-i}) \\ &\quad - (n-1)\sigma_v(\underline{\theta}). \end{aligned} \tag{3.7}$$

We may write

$$\sigma_{v,-i}(\underline{\theta}) = \sum_{j\neq i}\theta_j\log(1+a_{-i,j}(\underline{\theta}_{-i})),$$

so that when n is large, $\sigma_{v,-i}$ takes the value of σ_v, but with $n-1$ agents, i.e.,

$$\sigma_{v,-i} = \frac{n-1}{4}\left[-2\log\lambda(n-1)-1+\lambda^2(n-1)\right], \tag{3.8}$$

which will be slightly less than σ_v for n agents. Substitution of (3.8) into (3.7) yields

$$\begin{aligned} p_{VCG} \;\simeq\; & n\left(\frac{n-1}{4}\right)\left[-2\log\lambda(n-1)-1+\lambda^2(n-1)\right] \\ & -(n-1)\left(\frac{n}{4}\right)\left[-2\log\lambda(n)-1+\lambda^2(n)\right], \end{aligned}$$

which simplifies to the given expression for the mean-field approximation for p_{VCG}. A straightforward calculation of the limit as $n \to \infty$ proves the last assertion. $\square$

In Table 3.2, the mean-field approximation of $p_{VCG}(\underline{\theta})$ in Claim 2 is compared with the peak location of the histogram, for different number of agents. We remark that this approximation is remarkably accurate even for $n = 20$.

n	p_{VCG}	Peak in Figure 3.3
20	0.7269	0.7162
50	0.8182	0.8233
200	0.9048	0.9021
500	0.9387	0.9371

Table 3.2: Comparing $p_{VCG}(\underline{\theta})$ peak in histogram to the values obtained from Claim 2

From Claims 1 and 2, we can conclude that for large n, randomly (uniformly) chosen values of $\underline{\theta}$ will result in nearly constant values of $p_{VCG}(\underline{\theta})$ and $\sigma_v(\underline{\theta})$, so that each θ_i appears as linear coefficients in the constraints for optimization problem in (3.5). It can be shown using Lemma 1 proved by Guo & Conitzer [15] that these result in linear constraints on $c_2, c_3, \cdots, c_{n-1}, L(n)$. Additional to these constraints, linear constraints on $c_2, c_3, \cdots, c_{n-1}, L(n)$ obtained from e_k profiles are also included, and we have relaxation of problem in (3.5) to a simpler linear program. Of course, implementation of this simpler solution on a specific realization may either violate the F constraint or the solution may not be worst case optimal, hence the term relaxation; see Chapter 5. However, the (Lebesgue) measure of the set of profiles for which violation occurs vanishes as $n \to \infty$.

3.2 Optimal-in-expectation (OIE) Mechanism

In some scenarios, the worst-case $\underline{\theta}$ profiles may not occur very often. One may wish to minimize the efficiency loss in an expected sense. Thus, we design another mechanism in the class of Groves mechanisms which is optimal in expectation. The prior distribution over the agents' preferences are assumed to be known and the objective is to minimize the expected efficiency loss given by

$$\frac{E\left[p_{VCG}(\underline{\theta}) - \sum_{i=1}^{n} r_i(\underline{\theta}_{-i})\right]}{E\left[\sum_{i\in S} v_i(a_i^*(\underline{\theta}), \theta_i)\right]}, \tag{3.9}$$

subject to the same constraints (F) and (VP) as in the worst-case problem. By using the same form of linear rebate function as proposed above, the objective function becomes (with variables $c_2, \cdots, c_{n-1}$)

$$\frac{E\left[p_{VCG}(\underline{\theta})\right] - E\left[\sum_{i=2}^{N-1} c_i(i\theta_{i+1} + (N-i)\theta_i)\right]}{E\left[\sigma_v(\underline{\theta})\right]}. \tag{3.10}$$

Given prior distributions, the quantities $E[\theta_i]$, $E[\sigma_v(\underline{\theta})]$ and $E[p_{VCG}(\underline{\theta})]$ are constants. Thus the problem becomes

$$\max_{c_2,\cdots,c_{n-1}} \quad \sum_{i=2}^{n-1} c_i(iE[\theta_{i+1}] + (n-i)E[\theta_i]), \tag{3.11}$$

subject to

1. $\sum_{i=2}^{n-1} c_i(i\theta_{i+1} + (n-i)\theta_i) \leq p_{VCG}(\underline{\theta}), \ \forall \ \underline{\theta},$

2. $\sum_{i=2}^{k} c_i \geq 0,\ \forall\, k = 2, 3, \ldots, n-1.$

In the convex optimization problem (3.11), constraint 1) is the same as in the worst-case problem. As done for that problem, an approximate feasible region can be obtained via sampling. The problem can then be solved numerically to obtain the optimal linear rebate function coefficients. Simulation results are discussed in Chapter 5.

CHAPTER 4

Scalar Strategy, Efficient and Almost Budget Balanced Mechanisms

We now consider the case when the valuations functions are private information of the agents. In the case of divisible goods, the valuation function of agents is infinitely dimensional. Therefore, mechanisms for allocation of divisible goods, based only on scalar bids from agents, are of interest in many practical applications. Each agent reports a scalar value that is used to choose a surrogate valuation function from a single parameter family of valuation functions as in [19]. As the true valuation functions are unknown to the social planner, dominant strategy implementation is not possible. Instead, an efficient Nash equilibrium implementation, that is almost budget balanced, can be achieved.

Let $V_i(a_i)$ be the valuation for agent i when a_i is allocated, where $V_i : [0, \infty) \rightarrow R$ is concave, strictly increasing, and differentiable on $(0, \infty)$. An efficient allocation is a solution to the following problem:

$$\max_{\underline{a} \in A} \sum_{i \in N} V_i(a_i) \tag{4.1}$$

where A is a compact and convex set. Let the efficient allocation be $\underline{a}^v$.

Each agent sends a one-dimensional bid θ_i to the social planner. From the reported bids, the central planner constructs a surrogate valuation function $v_i^s(a_i, \theta_i)$, where $v_i^s(\cdot, \cdot)$ is as follows [19]:

(i) for every $\theta > 0$, $v_i^s(\cdot, \theta)$ is strictly concave, strictly increasing, continuous, and differentiable in $(0, \infty)$,

(ii) for every $\gamma \in (0, \infty)$ and $a \geq 0$, there exists a $\theta > 0$ such that $v_i^{s'}(a, \theta) = \gamma$, where $v_i^{s'}(a, \theta)$ is the derivative of $v_i^s(a, \theta)$ with respect to a.

The allocation and payment are calculated according to VCG mechanism, but using the surrogate valuation functions. These mechanisms are generally referred to as scalar strategy VCG (SSVCG) mechanisms [19]. A special case is the VCG-Kelly mechanism introduced in [22] where $v_i^s(a_i, \theta_i) = \theta_i f_i(a_i)$ for agent i, and the f_i's are strictly increasing, concave, and twice differentiable over $(0, \infty)$. In our mechanism, we include a rebate function as in Chapter 3 to obtain an almost budget balanced mechanism.

Let us represent the optimal allocation using surrogate valuation functions by

$$\underline{a}^s = \arg\max_{\underline{a} \in A} \sum_{i \in N} v_i^s(a_i, \theta_i),$$

where the dependence of $\underline{a}^s$ on $\underline{\theta}$ is understood and suppressed. The payment of i^{th} agent after rebate is denoted as

$$\begin{aligned} p_i^s(\underline{v}^s, \underline{a}^s) &= \sum_{j \neq i} v_j^s(\underline{a}^s_{-i}, \theta_j) - \sum_{j \neq i} v_j^s(\underline{a}^s, \theta_j) - r_i(\underline{\theta}_{-i}) \\ &= h_i(\underline{\theta}_{-i}) - \sum_{j \neq i} v_j^s(\underline{a}^s, \theta_j) \end{aligned}$$

where

$$\underline{a}^s_{-i} = \arg\max_{\underline{a}_{-i} \in A_{-i}} \sum_{j \in N, j \neq i} v_j^s(a_j, \theta_j).$$

The actual utility obtained by agent i is

$$\begin{aligned} u_i(\theta_i, \underline{\theta}_{-i}) &= V_i(a_i^s) - p_i^s(\underline{v}^s, \underline{a}^s) \\ &= V_i(a_i^s) + \sum_{j \neq i} v_j^s(a_j^s, \theta_j) - h_i(\underline{\theta}_{-i}). \end{aligned}$$

Finally, the bid vector $\underline{\theta}^{NE}$ is a Nash equilibrium if and only if

$$u_i(\theta_i^{NE}, \underline{\theta}_{-i}^{NE}) \geq u_i(\theta_i, \underline{\theta}_{-i}^{NE}), \ \forall \ \theta_i, \ \forall \ i \in N.$$

Johari & Tsitsiklis [19, Lem. 2] showed that, for any SSVCG mechanism, the bid vector $\underline{\theta}$ is a Nash equilibrium if and only if the corresponding $\underline{a}^s$, which implicitly depends on $\underline{\theta}$, satisfies

$$\underline{a}^s \in \arg\max_{\underline{a} \in A} \quad V_i(a_i) + \sum_{j \neq i} v_j^s(a_j, \theta_j) \ \forall \ i \in N.$$

Indeed, this result holds even for our proposed mechanism with rebates because even when the $h_i(\cdot)$ includes rebates it remains independent of the value reported by agent i. Further, [19, Cor. 3] states the existence of an efficient Nash equilibrium determined as follows. Agent i chooses θ_i such that $v_i^{s'}(a_i^v, \theta_i) = V_i'(a_i^v)$, i.e., each agents chooses her bid so that the declared marginal utility equals the true marginal utility. The resulting allocation satisfies $\underline{a}^s = \underline{a}^v$. Therefore, the resulting $\underline{\theta}$ is an efficient Nash equilibrium point. Thus, by using the rebate functions proposed in Chapter 3, we will obtain an almost budget balanced and efficient Nash equilibrium point.

CHAPTER 5

Simulation Setup and Results

5.1 Simulation setup

Simulation are done for allocation of a single perfectly divisible good to a number of agents each having the same valuation function $v_i = \theta_i \log(1 + a_i)$. The set A is defined by the set of all allocation vectors that satisfy $\sum_i a_i = 1$. The number of agents are varied from 3 to 50. The worst-case efficiency loss $(L(n))$ and coefficients $c_2, c_3, \dots, c_{n-1}$ are obtained by solving the optimization problem numerically over the approximate feasible region obtained using 5000 random generated $\underline{\theta}$'s and $\underline{e_k}$ profiles. For the optimal-in-expectation mechanism, the feasibility region is obtained in an analogous fashion with the modified objective function. Since $\underline{\theta}$ is uniformly distributed on Θ^N and then subsequently ordered, the ordered quantities satisfy $E[\theta_i] = \frac{n-i+1}{n+1}$, $i = 1, 2, \dots, N$. The coefficients $c_2, c_3, \dots, c_{n-1}$ are obtained by solving the optimization problem on the approximate feasible region. Efficiency losses for 200,000 θ profiles are calculated under these coefficients. The expected efficiency loss is obtained by taking the sample mean. The worst efficiency loss for optimal-in-expectation mechanism is obtained by calculating the efficiency losses for the $\underline{e_k}$ profiles and 200,000 uniformly sampled $\underline{\theta}$ profiles followed by an identification of the worst among them. After obtaining the optimal coefficients for both cases no constraint violations were observed by testing for 200,000 random generated $\underline{\theta}$ profiles which shows closeness of the approximate feasible region to actual one.

5.2 Simulation results

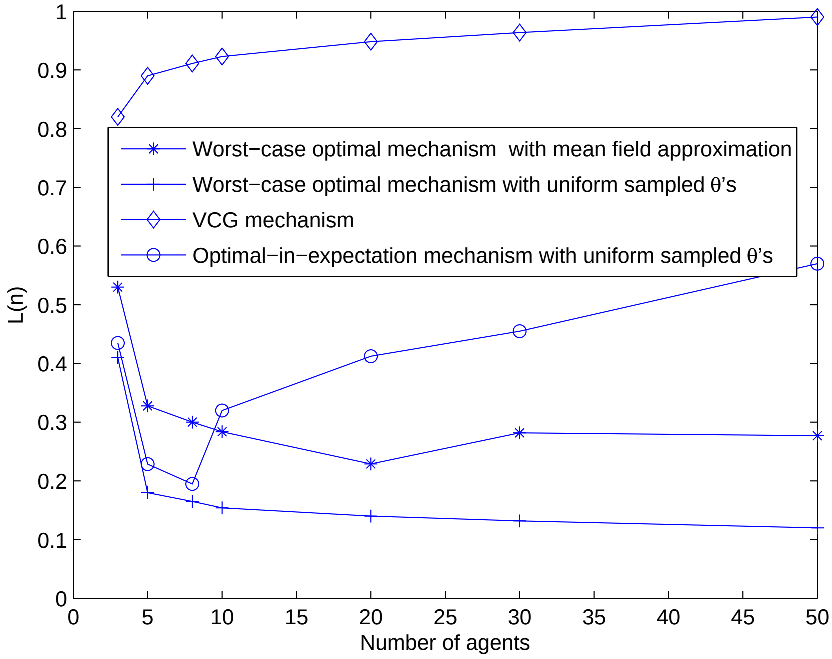

Figure 5.1: Worst-case efficiency loss of worst-case optimal, optimal-in-expectation and VCG mechanisms

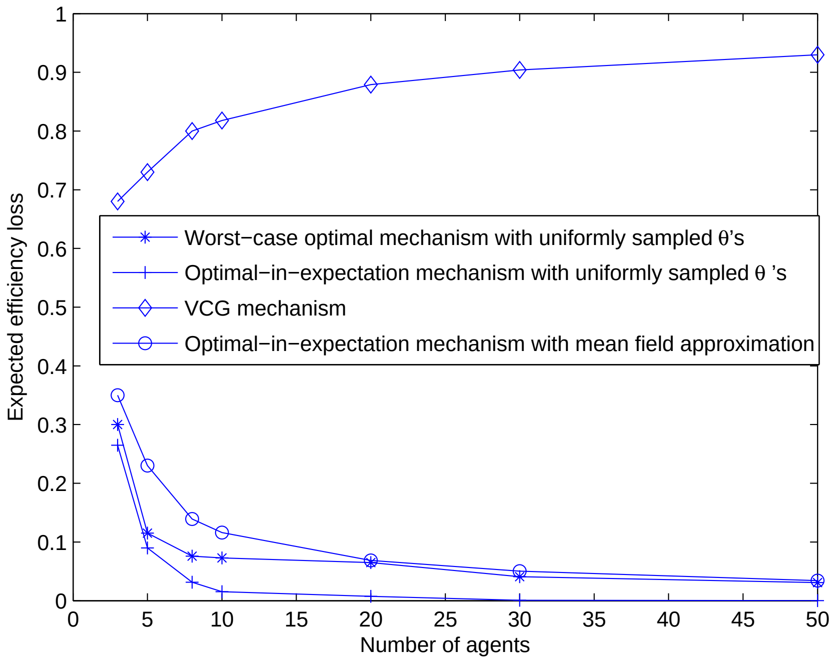

Figure 5.2: Expected efficiency loss of optimal-in-expectation, worst-case optimal and VCG mechanisms

In Figure 5.1, the worst-case optimal mechanism is compared with mean-field approximation mechanism which is explained in in Section 3.1.1, VCG mechanism and optimal-in-expectation mechanism for worst-case efficiency loss. It is observed that as number of agents increases the worst-case efficiency loss reduces for the worst-case optimal mechanism. On the other hand, the worst-case efficiency loss converges to 1 for the VCG mechanism. As expected, the optimal-in-expectation performs poorly in the worst-case sense when compared with worst-case optimal mechanism, especially for large number of agents. It can be observed that with mean-field approximation the resulting mechanism is not worst case optimal. In Figure 5.2, the expected efficiency loss of the optimal-in-expectation mechanism obtained by uniform sampling of $\underline{\theta}$ and mean-field approximation is compared with the worst-case optimal and VCG mechanisms. It can be seen from Figure 5.2 that the optimal-in-expectation mechanism obtained by uniform sampling of $\underline{\theta}$ outperforms the other three mechanisms in the expectation sense. The expected efficiency loss of the optimal-in-expectation and worst-case optimal mechanisms reduce as the number of agents increases. On the other hand, the expected efficiency loss of the VCG mechanism increases as the number of agents increase. In Figure 5.3, the fraction of violations of constraints and the worst-case ratio of the subsidy amount, i.e., total VCG payment minus sum of rebates and total VCG payment, are shown for the mean-field relaxation of (3.5). The fraction of violations of constraints and the absolute value of worst-case fraction of subsidy are observed to decrease with the number of agents.

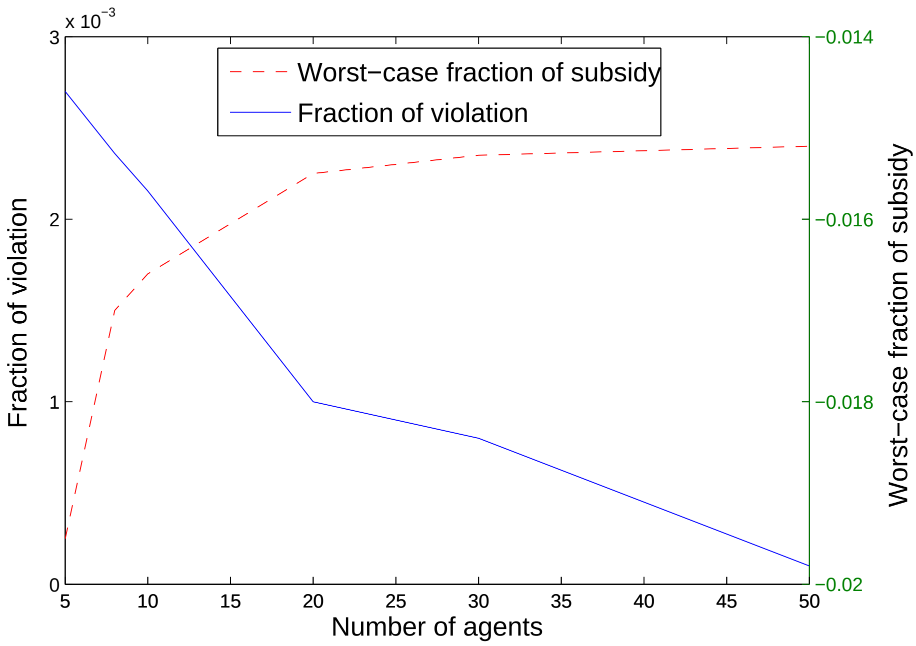

Figure 5.3: Mean-field relaxation: fraction of violation and worst-case fraction of amount of subsidy

CHAPTER 6

Discussion

In this chapter, we discuss the possibility of (i) inefficient mechanisms with more redistribution for divisible resource allocation, and (ii) mechanisms without money for divisible resource allocation.

6.1 Inefficient allocation mechanisms

The Groves class of mechanisms allocates resources efficiently but cannot be made budget balanced. Guo & Conitzer [23] proposed linear allocation mechanisms outside the Groves class for homogeneous indivisible goods allocation with unit demand. These mechanisms achieve better redistribution by inefficient allocation. The new definition of worst efficiency loss used for comparing inefficient mechanisms is

$$L(n) = \max_{\theta} \frac{\sigma_v(\underline{\theta}) - \sum_{i=1}^{n} v_i(a'_i, \theta_i) + \sum_{i=1}^{n} p_i(\underline{\theta})}{\sigma_v(\underline{\theta})}$$

where, $\underline{a}'$ is the inefficient allocation vector. Note that this quantity will be equal to the earlier definition of $L(n)$ for the case of allocatively efficient mechanisms. Two ways of achieving better redistribution by inefficient allocation have been proposed – partitioning and burning. These methods could be considered for divisible allocation as follows.

6.1.1 Partitioning agents and resource

Partition the agents into groups of size n_1 and $n - n_1$. A fraction p ($0 \leq p \leq 1$) part of the resource is allocated among the n_1 agents according to VCG mechanism. Likewise, remaining $1 - p$ part of the resource is allocated among the other group of $n - n_1$ agents. The VCG payments are completely transferred between the groups in a predetermined way. Thus, the mechanism becomes strongly budget balanced. However, there will be an efficiency loss due to the initial inefficient allocation (partitioning).

6.1.2 Burning a part

Consider a mechanism where $1 - p$ part of the resource is burnt. The remaining part is optimally allocated to all the n agents. Burning can be considered as a special case of partitioning where $n_1 = n$. In [23], burning some of the goods among the homogeneous goods is observed to be better in terms of worst-case efficiency loss in some cases.

We observed from simulations of the above methods in the case of infinitely divisible resources, that the worst-case efficiency loss did not improve compared to worst-case optimal and efficient mechanisms proposed in Section 3. The improvement achieved in [23] seems to depend on the indivisible nature of the goods and the unit demand requirement. Further analysis of these mechanisms would be interesting, but is outside the scope of our present work.

6.2 Mechanisms without money

Thus far, we assumed that money is transferred as payments by agents. But, using money as payments require complex billing mechanism. In certain problems, such as the resource allocation problem in the uplink of a multiple access channel, one desires mechanisms without money transfer where the resource allocation itself is adjusted in lieu of payments. In such cases, the allocation to the agents after adjusting for payments and rebates should fall in the feasible allocation set. Thus, there is an additional constraint in the optimization problem while designing rebates.

Table 6.1: Comparing $L(n)$ of VCG mechanism and Worst Case Optimal (WCO) mechanism without money for $n = 20$

	VCG mechanism	WCO mechanism
$\alpha = 1$	0.958	0.919
$\alpha = 0.5$	0.963	0.955
$\alpha = 0$	1	1

For simulations we considered α-utility valuation function, i.e.,

$$v_i(a_i;\theta_i) = \begin{cases} \theta_i \frac{(1+a_i)^{(1-\alpha)}-1}{1-\alpha}, & \text{if } \alpha \geq 0 \text{ and } \alpha \neq 1 \\ \theta_i \log(1+a_i), & \text{if } \alpha = 1 \end{cases}$$

and $\sum_i a_i = 1$. The $L(n)$ values for VCG mechanism and Worst case optimal mechanism without money for $n = 20$ are given in Table 6.1 above for different values of α. Also, the expected efficiency loss values for VCG mechanism and Optimal-in-expectation mechanism without money for $n = 20$ are compared in Table 6.2 above. It is observed that for all values of α there is not much improvement in efficiency loss from VCG mechanism.

Table 6.2: Comparing Expected efficiency loss of VCG mechanism and Optimal-in-Expectation (OIE) mechanism without money for $n = 20$

	VCG mechanism	OIE mechanism
$\alpha = 1$	0.879	0.849
$\alpha = 0.5$	0.92	0.8905
$\alpha = 0$	0.95	0.9021

CHAPTER 7

Conclusions

7.1 Conclusions

Efficient resource allocation is achieved in the uplink of a cellular system by making the users to report true queue information while minimizing the loss due to payments. For this, we proposed mechanisms for allocation of a single divisible resource to a number of agents when the agents report only scalar values. We proposed a mechanism in the class of Groves mechanisms that is almost budget balanced as it minimizes the worst-case efficiency loss. The proposed mechanism is feasible and has voluntary participation and anonymity properties. The mechanism is applicable to allocation of divisible or indivisible goods and simplifies to the mechanism proposed by Moulin [16] and Guo & Conitzer [15] and Gujar & Narahari [18] for the indivisible goods case in the respective special settings. A mechanism that is optimal-in-expectation is also proposed by assuming that the distribution of the preferences are known. All the mechanisms are designed in a convex optimization framework. The convex optimization problems were numerically solved to obtain the optimal coefficients of the linear rebate function. The solution is obtained over an approximate feasible region via sampling of constraints. We also obtained a mean field approximation of the problems and linear program relaxation of the problems were solved.

The proposed worst-case optimal and optimal-in-expectation mechanisms are compared with each other and with the VCG mechanism, in both worst-case and optimal-in-expectation senses. A significant reduction in efficiency loss is obtained for both linear rebate mechanisms when compared to the VCG mechanism. Also, as number of agents increases the efficiency loss tends to zero for the proposed mechanisms in the respective sense.

We also discussed extensions of our proposed mechanisms to a case where the valuation functions are private information to agents. The agents report only scalar values and surrogate valuation functions are constructed from them (Johari & Tsitsiklis in [19]). A similar optimization will yield almost budget balanced and efficient Nash equilibrium implementation for this setting. Mechanisms outside Groves class that are more competitive but inefficient were proposed in [23]. Issues related to their extensions to the divisible case were also discussed. Issues while implementing mechanisms without money were also discussed for different possible valuation functions.

7.2 Future work

In this work, we restricted ourselves to linear redistribution mechanisms. We believe that there can be nonlinear redistribution mechanisms which are more budget balanced than linear ones for divisible resource allocation. Therefore, investigating possible nonlinear redistribution mechanisms will be an interesting future direction. Another possible extension is to look for more budget balanced but allocatively inefficient and/or non strategy proof mechanisms. The design of mechanisms without money[28] that achieve lower efficiency loss is also a possible area of future research.

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Publications Related To Thesis

1. A.K. Chorppath, S. Bhashyam and R.Sundaresan,"Almost budget balanced mechanism for divisible resource allocation," *proceedings of* 47^{th} *Annual Allerton Conference on Communication, Control and Computing,* Monticello, USA, September 2009.

2. A.K. Chorppath, S. Bhashyam and R.Sundaresan,"A convex optimization framework for almost budget balanced allocation of a divisible good,"*submitted to IEEE Transactions on Automation Science and Engineering*, December 2009.