LDPC codes for OFDM over an Inter-symbol Interference Channel

Dileep M. K.
Srikrishna Bhashyam
Andrew Thangaraj

Department of Electrical Engineering
IIT Madras

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Background on LDPC codes

- Low Density Parity Check (LDPC) codes
  - linear codes with sparse parity-check matrices
  - simple definition, capacity-approaching performance
- LDPC analysis and design
  - large ensembles of codes - all with same performance
  - random code from ensemble performs close to average
- How is the ensemble specified?
  - weights of the columns and rows of the parity-check matrix
  - weights are collected into weight distribution polynomials

Analysis and Design Tools for LDPC Codes

Study average performance of ensemble of codes whose parity-check matrices have the same weight distribution

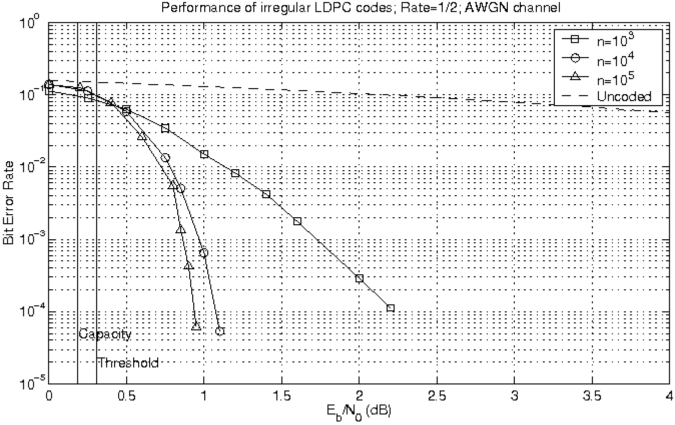
Message-Passing Decoders, Thresholds, Density Evolution

- Message-passing decoders: practical, iterative
  - performance of ensemble is studied under message-passing decoding
- Threshold phenomenon
  - threshold $=$ SNR* $\Rightarrow$ SNR $>$ SNR* will result in successful decoding
  - block-length $\rightarrow \infty$, iterations $\rightarrow \infty$
- Density evolution
  - tool to determine threshold

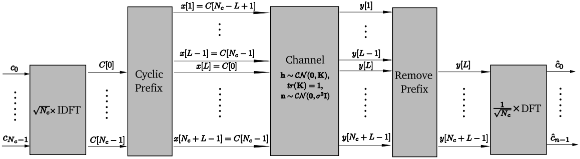
Study of LDPC codes in a new system involves...

developing a density evolution algorithm and determination of threshold

Threshold phenomenon



OFDM



- The channel model

$$\hat{\mathbf{c}} = \mathbf{H}.\mathbf{c} + \mathbf{N}.$$

- Binary Input alphabet. BPSK modulation.
- Assumptions:
  - A codeword is distributed over a single OFDM symbol
  - The blocklength of the code (N_c) tend to infinity
- In the limit, there is no cyclic prefix overhead

Prior Work on LDPC Codes in an OFDM System

- Prior work on LDPC over OFDM
  - Mannoni *et al*: mixture PDF analysis and optimization of degree distribution
  - Baynast *et al*: positioning of information bits in OFDM subcarriers
- Prior work on LDPC over ISI
  - Kavcic *et al*: LDPC codes over binary-input ISI channels with BCJR
- Previous theoretical works employ a Gaussian mixture density analysis for threshold estimation
- No rigorous proof for the existence of threshold in OFDM systems

Our Work

- Propose a rigorous density evolution
- Existence of LDPC thresholds
- Method for threshold estimation
- Comparison of LDPC thresholds with OFDM capacity
- Comparisons between the time-domain BCJR algorithm proposed by Kavciv *et al*
- Mercury/Waterfiling power allocation to improve the OFDM capacity and LDPC thresholds

LDPC Codes : Regular and Irregular

- Regular LDPC Codes
  - H matrix with constant column weight (w_c) and constant row weight (w_r)
  - Notation : (n,w_c,w_r) regular code
- Irregular LDPC Codes
  - Column weights (row weights) are not equal
  - Bit node degree distribution $\lambda(x) = \sum_{i=2}^{d_v} \lambda_i x^{i-1}$
    λ_i : the fraction of all edges connected to variable nodes of degree i
  - Check node degree distribution $\rho(x) = \sum_{j=2}^{d_c} \rho_j x^{j-1}$
    ρ_j : the fraction of all edges connected to check nodes of degree j
  - Notation: (n, λ, ρ)

Density Evolution

- Tracks the evolution of the pdf of the messages
- Initial Message: LLR of the received value

  For AWGN channel, initial PDF of the messages $f_0 \equiv \mathcal{N}\left(\frac{2}{\sigma^2}, \frac{4}{\sigma^2}\right)$
- PDF of the messages after l rounds of message passing is calculated recursively

$$f_l = f_0 \otimes \lambda\left(\rho\left(f_{l-1}\right)\right)$$

$$\lambda(f) := \sum_i \lambda_i f^{\otimes(i-1)}, \qquad \rho(f) := \sum_i \rho_i f^{\boxtimes(i-1)}$$

- Average probability of error after l^{th} iteration at given SNR:

$$Pr(error)^l = Pr(message < 0) + \frac{1}{2} Pr(message = 0)$$

Density Evolution: Conditions

- Channel Symmetry

$$p(y_t = q|x_t = 1) = p(y_t = -q|x_t = -1).$$

- Decoder Symmetry
  - Variable node symmetry
  - Check node symmetry
- Advantage: Error probability becomes independent of codeword
- Symmetry of Message PDF

$$f_l(x) = e^x f_l(-x).$$

LDPC over OFDM : Symmetry conditions

- Channel Symmetry
  - OFDM channel → Parallel AWGN channels
  - Each channel is symmetric.

$$p_{Z_i|C_i}(z_i|c_i = 1) = p_{Z_i|C_i}(-z_i|c_i = -1).$$

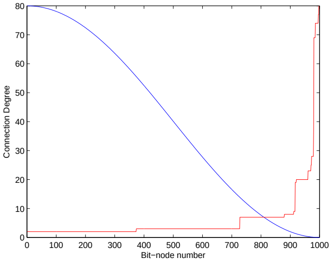
- Analysis can be restricted to the All-one Codeword
- LLR density in the ith channel:

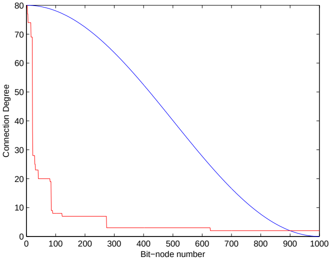
$$U_i \sim \mathcal{N}\left(\frac{4|H[i]|^2}{\sigma^2}, \frac{8|H[i]|^2}{\sigma^2}\right).$$

- LLR distribution is symmetric

Interleaving

- How should the bits be assigned to the subcarriers?



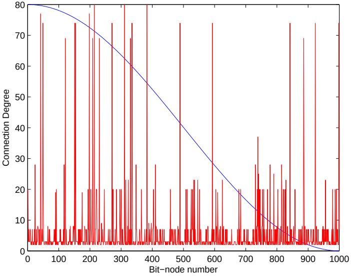


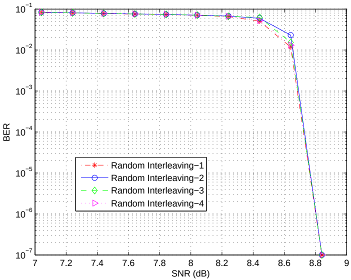
- Equivalent to the design of an interleaver
- Is there an optimum assignment?.
- Are we going to analyze the LDPC performance for a *given assignment*?
  - Gaussian approximation is necessary in the analysis

Random Interleaving

- Concentration Theorem:

  LDPC performance with different random interleaving are concentrated around the average performance
- It is enough to analyze this average performance
- Eliminates the need for Gaussian approximation in the analysis





Concentration Theorem

- LDPC performance with different random interleaving are concentrated around the average performance
- Define: $p^{l}_{\underline{H_i}}$ = probability of incorrect message along an edge at the lth iteration when the interlever chosen uniformly at random is $\underline{H_i}$.
- Define: Error concentration probability $\overline{p} = \frac{1}{N!}\sum_{i=1}^{N!} p^{l}_{\underline{H_i}}$
- Theorem:

$$P\left(\left|p_{\underline{H_i}} - \overline{p}\right| \geq \frac{\epsilon}{2}\right) \leq 2e^{-\beta\epsilon^2 n}.$$

- It is enough to analyze this average performance
- Eliminates the need for Gaussian approximation in the analysis

Initial PDF estimation and Density Evolution Algorithm

The algorithm

Consider a degree distribution pair (λ, ρ) *and transmission over an OFDM channel with* N_c *subcarriers with code of blocklength* $n = N_c$*, with associated L-densities* $\tilde{f}_i, i \in \{1, 2, \ldots, N_c\}$ *. Define*

$$f_0 = \frac{1}{N_c} \sum_{i=1}^{N_c} \tilde{f}_i,$$

then for $l \geq 1$*,*

$$f_l = f_0 \otimes \lambda\left(\rho\left(f_{l-1}\right)\right),$$

- Monotonicity and Threshold
  - The update equations : Same as AWGN
  - Same monotonicity argument
  - Existence of threshold!!

Threshold Estimation

- We let the number of subcarriers N_c tend to infinity
- LLR distribution depends on the the DTFT of the channel impulse response $H(e^{j\omega})$

$$f(u,\omega) = \frac{\sigma}{4|H(e^{j\omega})|\sqrt{\pi}} \exp\left[-\frac{(\sigma^2 u - 4|H(e^{j\omega})|^2)^2}{16|H(e^{j\omega})|^2\sigma^2}\right]$$

$$H(e^{j\omega}) = \sum_{i=-\infty}^{\infty} h[i]e^{-j\omega i}$$

- LLR distribution is now a continuous function of the angular frequency ω
- Summation changes to an integral

$$f_0(u) = \frac{1}{2\pi}\int_0^{2\pi} f(u,\omega).d\omega$$

Threshold Estimation: Channel with spectral nulls

- The function $f(u, \omega)$ is not always well behaved
- Problems in channels with spectral nulls
- New approach to calculate the $f_0(u)$
- Using the idea of characteristic function

$$\begin{array}{ccc} f(u,\omega) & \nrightarrow & f_0(u) \\ \downarrow & & \uparrow \\ \hat{f}(t,\omega) & \rightarrow & \hat{f}(t) \end{array}$$

Threshold Estimation

- Characteristic function:

$$\begin{aligned}\hat{f}(t,\omega) &:= \int_{-\infty}^{\infty} f(u,\omega)e^{jut}du \\ &= \exp\left[-\frac{4|H(e^{j\omega})|^2 t^2}{\sigma^2} + j\frac{4|H(e^{j\omega})|^2 t}{\sigma^2}\right]\end{aligned}$$

- Advantage: A well behaved characteristic function obtained analytically

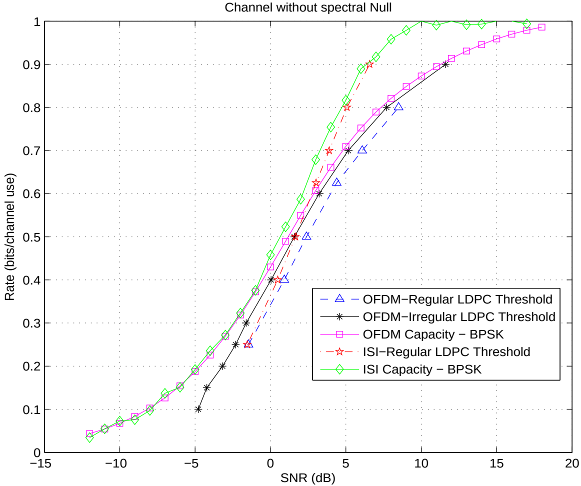
$$\begin{aligned}\hat{f}(t) &:= \frac{1}{2\pi}\int_{0}^{2\pi} \hat{f}(t,\omega)\ d\omega \\ f_0(u) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(t)\ e^{-jut}\ dt\end{aligned}$$

Results

- Thresholds for different rate regular and irregular LDPC codes
- Validation by simulation
- Comparison with OFDM capacity
- Comparison with LDPC threshold over a binary ISI channel with BCJR equalization

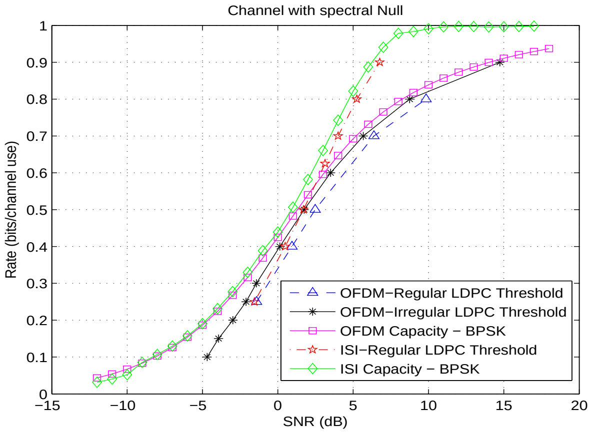
Thresholds: Channel without spectral null

- Channel: $\{h_2[i]\} = [0.800, 0.600]$



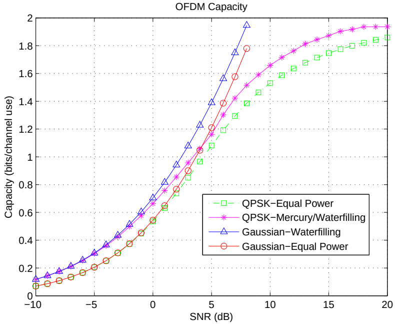
Thresholds: Channel with spectral null

- Channel: $\{h_1[i]\} = \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$.



Mercury/Waterfilling Power allocation

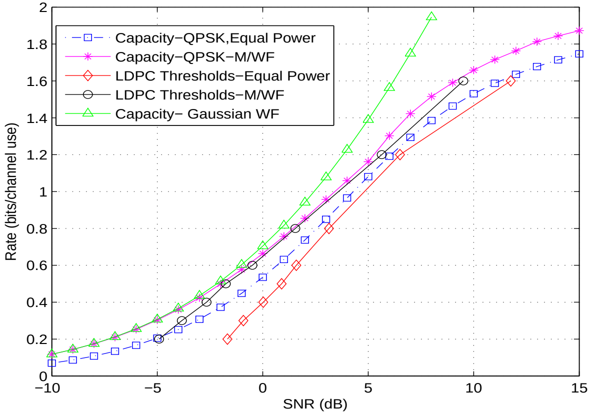
- The optimum power allocation for parallel Gaussian channels with arbitrary input constellation
- Channel: $\{h_1[i]\} = \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$.



- Applying Mercury/waterfilling for better LDPC thresholds

LDPC thresholds with Mercury/Waterfilling Power allocation

- Channel: $\{h_1[i]\} = \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}.$



Conclusions

- Developed a rigorous density evolution for binary-input OFDM and proved the existence of thresholds
    - LDPC thresholds are very close to OFDM capacity at higher rates
- Compared OFDM-BPSK capacity and ISI-BPSK capacity
    - At higher rates, ISI-LDPC thresholds are much better than OFDM-LDPC thresholds
- Mercury/Waterfilling power allocation over OFDM subcarriers
    - Again, LDPC thresholds are very close to OFDM capacity

Future work

- Achieving capacity at very low rates
- Optimum bit-loading with Mercury/Waterfilling power allocation to improve capacity and thresholds
- Optimization of irregular LDPC code for OFDM
- Extension to wireless channels

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