SOLUTIONS TO ASSIGNMENT - 8

1. Differential Entropy

(a) Exponential distribution

$$h(f) = -\int_0^\infty \lambda e^{-\lambda x}(\ln\lambda - \lambda x)\,dx = -\ln\lambda + 1 \text{ nats}$$

$$= \log\frac{e}{\lambda} \text{ bits}$$

(b) Laplace density

$$h(f) = -\int_{-\infty}^{\infty} \frac{1}{2}\lambda e^{-\lambda|x|}\left[\ln\frac{1}{2} + \ln\lambda - \lambda|x|\right]dx$$

(even fn)

$$= -\ln\frac{1}{2} - \ln\lambda + 1$$

$$= \ln\frac{2e}{\lambda} \text{ nats} = \log\frac{2e}{\lambda} \text{ bits}$$

(c) Sum of two normal distributions.

Now $X_1 + X_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

$$h(f) = \frac{1}{2}\log 2\pi e(\sigma_1^2+\sigma_2^2) \text{ bits}$$

2) $X_1 \sim N(0, K_1) \quad X_2 \sim N(0, K_2)$

$\Pr(\theta = 1) = \lambda \quad \Pr(\theta = 2) = \bar{\lambda}$

Now let $\boxed{Z = X_\theta} \Rightarrow K_Z = \lambda K_1 + \bar{\lambda} K_2$

Now $h(Z) \le \frac{1}{2} \log (2\pi e)^n |\lambda K_1 + \bar{\lambda} K_2|$

(Gaussian maximises entropy)

$\Rightarrow$ Z is not multivariate normal.

But $h(Z) \ge h(Z|\theta) = \lambda h(X_1) + \bar{\lambda} h(X_2)$

$= \frac{\lambda}{2} \log (2\pi e)^n |K_1| + \frac{\bar{\lambda}}{2} \log (2\pi e)^n |K_2|$

$= \frac{1}{2} \log \left\{ (2\pi e)^n |K_1| \right\}^{\lambda} \left\{ (2\pi e)^n |K_2| \right\}^{\bar{\lambda}}$

$= \frac{1}{2} \log \left\{ (2\pi e)^n |K_1|^{\lambda} |K_2|^{\bar{\lambda}} \right\}$

Thus $\frac{1}{2} \log (2\pi e)^n |\lambda K_1 + \bar{\lambda} K_2| \ge h(Z) \ge \frac{1}{2} \log \left\{ (2\pi e)^n |K_1|^{\lambda} |K_2|^{\bar{\lambda}} \right\}$

$\Rightarrow |\lambda K_1 + \bar{\lambda} K_2| \ge |K_1|^{\lambda} |K_2|^{\bar{\lambda}}$

3) $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(0, \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}\right)$

$h(X, Y) = \frac{1}{2} \log (2\pi e)^2 |K| = \frac{1}{2} \log \left\{ (2\pi e)^2 \sigma^4 (1 - \rho^2) \right\}$

$X \sim N(0, \sigma^2)$ $\quad Y \sim N(0, \sigma^2)$

$\Rightarrow h(X) = h(Y) = \frac{1}{2} \log 2\pi e \sigma^2$

$I(X;Y) = h(X) + h(Y) - h(X,Y) = -\frac{1}{2} \log (1 - \rho^2)$

(a) $\rho = 1 \Rightarrow I(X;Y) = \infty$

Since $\rho = 1$ $X = Y$. Hence mutual information $\to \infty$.

(b) $\rho = 0$. $I(X;Y) = 0$ since X & Y are independent and hence $h(Y|X) = h(Y)$ so $I(X;Y) = 0$

(c) $\rho = -1 \Rightarrow X = -Y$. Again from (a) $I(X;Y) = \infty$. (sign doesn't matter)

4) $Y = X + Z$

$\Rightarrow$ distribution of Y = convolution of densities of X & Z (since they are independent)

for $a \le 1$

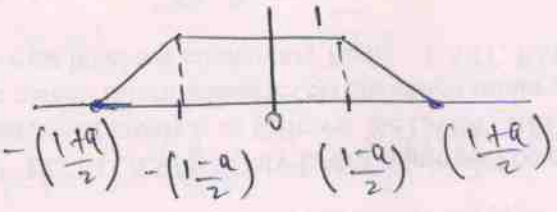
$$P_Y(y) = \begin{cases} \frac{1}{a}\left(y + \left(\frac{1+a}{2}\right)\right) & -\left(\frac{1+a}{2}\right) \le y \le -\left(\frac{1-a}{2}\right) \\ 1 & -\left(\frac{1-a}{2}\right) \le y \le \left(\frac{1-a}{2}\right) \\ \frac{1}{a}\left(-y + \left(\frac{1+a}{2}\right)\right) & \left(\frac{1-a}{2}\right) \le y \le \left(\frac{1+a}{2}\right) \end{cases}$$

and for $a \ge 1$

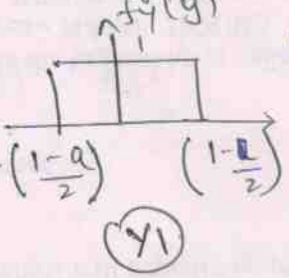
$$P_Y(y) = \begin{cases} \frac{1}{a}\left(y + \left(\frac{1+a}{2}\right)\right) & -\left(\frac{a+1}{2}\right) \le y \le -\left(\frac{a-1}{2}\right) \\ \frac{1}{a} & -\left(\frac{a-1}{2}\right) \le y \le \left(\frac{a-1}{2}\right) \\ \frac{1}{a}\left(-y + \left(\frac{1+a}{2}\right)\right) & \left(\frac{a-1}{2}\right) \le y \le \left(\frac{a+1}{2}\right) \end{cases}$$

$a = 1 \Rightarrow Y$ is triangular over $(-1, 1)$

Case 1) $a \le 1$.



$\Longrightarrow$ divide into two regions



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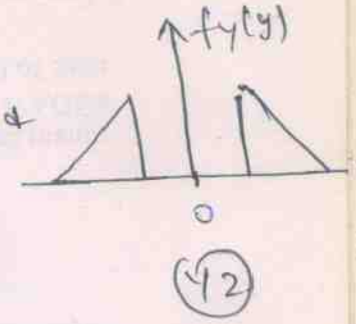


Now let
$$\lambda = \begin{cases} 1 & \text{if } Y = Y_1 \\ 2 & \text{if } Y = Y_2 \end{cases}$$
Thus $X = f(Y)$ $\Rightarrow h(X|Y) = 0$

Now
$$h(\lambda, Y) = h(Y) + h(\lambda|Y) = h(Y)$$
$$= h(\lambda) + h(Y|\lambda)$$

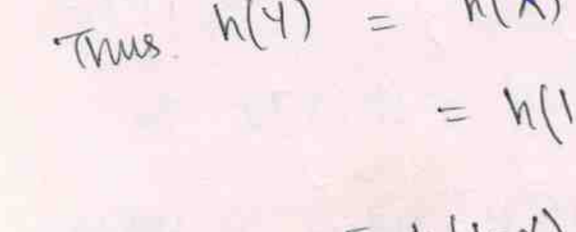
Thus
$$h(Y) = h(\lambda) + h(Y|\lambda)$$
$$= h(1-\alpha) + (1-\alpha)\, h(Y_1) + \alpha H(Y_2)$$
$$= h(1-\alpha) + (1-\alpha) \ln \frac{1}{1-\alpha} + \underbrace{\alpha\left(\frac{1}{2} + \ln a\right)}_{I,}$$

Triangular distribution

$$= \frac{\alpha}{2} \text{ nats}$$

Case 2) $a \ge 1$

Now $p(\lambda = 1) = \frac{a-1}{a}$ $\left\{= \Pr(Y_1)\right\}$



$$H(Y) = h(\lambda) + h(Y|\lambda)$$
$$= h\left(1 - \frac{1}{a}\right) + \left(\frac{a-1}{a}\right) \ln(a-1) + \frac{1}{a}\left(\frac{1}{2} + \ln 1\right)$$

$$= \frac{1}{a} \ln a + \left(\frac{a-1}{a}\right) \ln \frac{a}{a-1} + \frac{a-1}{a} \ln(a-1) + \frac{1}{2a}$$

$$= \ln a + \frac{1}{2a}$$

$$I(X;Y) = h(Y) - h(Y|X) = h(Y) - h(Z)$$

$$h(Z) = \ln a$$

Thus $I(X;Y) = \begin{cases} \frac{a}{2} - \ln a & a \le 1 \\ \frac{1}{2a} & a \ge 1 \end{cases}$

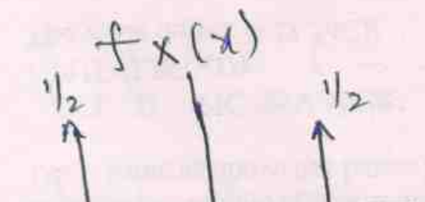
(b) $a = 1 \Rightarrow$ range of Y is $(-1, 1)$

$$I(X;Y) = h(Y) - h(Y|X) = h(Y) - h(Z)$$
$$= h(Y) - \ln 1 = h(Y)$$

$h(Y)$ is maximum when it is uniform over $(-1, 1)$

$$\Rightarrow I(X;Y) = h(Y) = \ln 2$$

what X makes Y uniform? The convolution of $f_X(x) * f_Z(z)$ is uniform.



5) Two look Gaussian:

The input distribution that maximizes capacity is

$X \sim N(0, P)$.

$$C = \max I(X; Y_1 Y_2)$$
$$= h(Y_1 Y_2) - h(Y_1 Y_2 | X) = h(Y_1 Y_2) - h(Z_1 Z_2 | X)$$
$$= h(Y_1 Y_2) - h(Z_1 Z_2) \quad (Z_1 Z_2 \text{ independent of } X)$$

$$h(Z_1 Z_2) = \frac{1}{2} \log (2\pi e)^2 |K_Z| = \frac{1}{2} \log (2\pi e)^2 N^2 (1 - \rho^2)$$

Now

$$(Y_1 Y_2) \sim N\left(0, \begin{bmatrix} P+N & P+\rho N \\ P+\rho N & P+N \end{bmatrix}\right)$$

$$h(Y_1 Y_2) = \frac{1}{2} \log (2\pi e)^2 \left(N^2 (1-\rho^2) + 2PN(1-\rho)\right)$$

Thus $C = \frac{1}{2} \log\left(1 + \frac{2P}{N(1+\rho)}\right)$

(a) $\rho = 1$ $\quad C = \frac{1}{2} \log\left(1 + \frac{P}{N}\right)$ since $Y_1 = Y_2$

(b) $\rho = 0$ $\quad C = \frac{1}{2} \log\left(1 + \frac{2P}{N}\right)$

(c) $\rho = -1$ $\quad C = \infty$. Since $Z_1 = -Z_2$ and adding $(Y_1 + Y_2)$ we can cancel out Noise.

6) Multipath Gaussian:

$$Y = (a_1 + a_2) X + Z_1 + Z_2$$

$$K_Z = \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix} \Rightarrow Z_1 + Z_2 \sim N(0, \sigma_Z^2)$$

$$\sigma_Z^2 = (\sigma^2 + \rho\sigma^2) + (\sigma^2 + \rho\sigma^2)$$

$$= 2\sigma^2(1+\rho).$$

Thus

$$C = \max I(X;Y) = \frac{1}{2}\log_2\left(1 + \frac{(a_1+a_2)^2 P}{2\sigma^2(1+\rho)}\right)$$

a) $\rho = 0 \Rightarrow C = \frac{1}{2}\log_2\left(1 + \frac{(a_1+a_2)^2 P}{2\sigma^2}\right)$

b) $\rho = 1 \Rightarrow C = \frac{1}{2}\log_2\left(1 + \frac{(a_1+a_2)^2 P}{4\sigma^2}\right)$ since $Z_1 = Z_2$

c) $\rho = -1 \Rightarrow C = \frac{1}{2}\log_2(1+\infty) = \infty$ since $Z_1 = -Z_2$ (cancel)

7) Parallel Gaussian channels:

Given $\beta_1 P_1 + \beta_2 P_2 \leq \beta$.

(a) Note $C_i = \frac{1}{2}\log\left(1 + \frac{P_i}{N_i}\right) = \frac{1}{2}\log\left(1 + \frac{\beta_i P_i}{\beta_i N_i}\right)$

New Noise powers: $\beta_1 N_1$ and $\beta_2 N_2$.

Ideal solution is waterfilling.

In waterfilling solution parallel channels stop acting like single channel when $|P| \geq |N_1 - N_2|$. Applying this here

$\beta \geq |\beta_1 N_1 - \beta_2 N_2|$

(b) Now $\beta_1 N_1 = 3 \quad \beta_2 N_2 = 4$

Using waterfilling $\beta_1 P_1 = 5.5 \quad \beta_2 P_2 = 4.5$

Thus optimum power $P_1 = 5.5 \quad P_2 = 2.25$

and $c = c_1 + c_2 = \frac{1}{2}\log\left(1 + \frac{5.5}{3}\right) + \frac{1}{2}\log\left(1 + \frac{2.25}{2}\right)$

$= 1.29$ bits