EE6340 - Information Theory

Problem Set 1 Solution

February 21, 2013

1. a) Random variable X= No. of coin tosses till the first head appears. If $\mathbb{P}(\text{head})=p$ and $\mathbb{P}(tail) = q$,

$$\begin{aligned}
\mathbb{P}(x = n) &= pq^{n-1} \\
\implies H(X) &= -\sum_{n=1}^{\infty} pq^{n-1}\log(pq^{n-1}) \\
&= -\frac{p\log p}{1-q} - \frac{pq\log q}{p^2} \\
&= H(p)/p
\end{aligned}$$

For $p = \frac{1}{2}$, $H(X) = H(0.5)/0.5 = 2$ bits.

b) Best questions are those which have equal chances of being answered as Yes or as No.
$\mathbb{P}(x = 1) = \frac{1}{2}$ and $\mathbb{P}(\text{x=2 or 3 or 4...})=\frac{1}{2}$ (equally probable).
So first question is, "Is it 1 or not?'".
Similarly,$\mathbb{P}(x = 2) = \frac{1}{4}$ and $\mathbb{P}(\text{x=3 or 4 or 5...})=\frac{1}{4}$ (equally probable)
So second question is, "Is it 2 or not?'".
The questions proceed as above.
$\mathbb{E}(\text{No.of questions}) = \sum_{n=1}^{\infty} n\frac{1}{2^n} = 2 = H(X)$.
In general,$\mathbb{E}(\text{No.of questions}) \geq H(X)$
This problem can be interpreted as a source coding problem with 0=no, 1=yes, X=Source, Y=Encoded sequence.

2. a) $H(X, g(X)) = H(X) + H(g(X)|X)$ (Chain rule)

b) $H(g(X)|X) = \sum_x p(x)H(g(X)|X = x) = \sum_x p(x)0 = 0$ (For a given x,g(x) is fixed).
$\implies H(X, g(X)) = H(X)$

c) $H(X, g(X)) = H(g(X)) + H(X|g(X))$ (Chain rule)

d) $H(X|g(X)) \geq 0$ with equality if $g(.)$ is one-to-one.
(a),(b) and (c) $\implies H(X, g(X)) \geq H(g(X)$

3. Let there be 2 y_i's y_1 and y_2 such that for $x = x_0$, $p(x_0, y_1) > 0$ and $p(x_0, y_2) > 0$.
$\implies p(y_1|x_0) > 0$ and $p(y_2|x_0) > 0$, neither of them being 0 or 1.

$$\begin{aligned}
H(Y|X) &= -\sum_{x,y} p(x, y)\log_2 p(y|x) \\
&\geq -p(x_0)p(y_1|x_0)\log_2 p(y_1|x_0) - p(x_0)p(y_2|x_0)\log_2 p(y_2|x_0) \\
&> 0
\end{aligned}$$

since $-t\log t > 0$ for $0 < t < 1$. So, H(Y|X)= 0 iff Y is a function of X. Else H(Y|X)>0.

4. X=Outcome of world series
Y=No.of games played ϵ {4,5,6,7}
For $Y=4$, there are 2 outcomes {AAAA,BBBB} each with probability $\frac{1}{2^4}$.
For $Y=5$,there are $2\times\binom{4}{3}=8$ outcomes each with probability $\frac{1}{2^5}$.
For $Y=6$,there are $2\times\binom{5}{3}=20$ outcomes each with probability $\frac{1}{2^6}$.
for $Y=7$,there are $2\times\binom{6}{3}=40$ outcomes each with probability $\frac{1}{2^7}$.
Thus,

$$\mathbb{P}(Y=4)=\frac{1}{8}$$
$$\mathbb{P}(Y=5)=\frac{1}{4}$$
$$\mathbb{P}(Y=6)=\frac{5}{16}$$
$$\mathbb{P}(Y=7)=\frac{5}{16}$$

X={AAAA,BBBB,....,BABABAB,....BBBAAAA}
$\mathbb{P}$(BAAAA)=$\frac{1}{2^5}$, $\mathbb{P}$(BABABAB)=$\frac{1}{2^7}$,...
There are 2 sequences of length 4, 8 sequences of length 5, 20 sequences of length 6 and 40 sequences of length 7.

$$\begin{aligned}H(X)=&-\sum_x p(x)\log_2 p(x)\\ =&2\left(\frac{1}{16}\right)\log_2 16+8\left(\frac{1}{32}\right)\log_2 32+20\left(\frac{1}{64}\right)\log_2 64+40\left(\frac{1}{128}\right)\log_2 128\\ =&5.8125\\ H(Y)=&-\sum_y p(y)\log_2 p(y)\\ =&\frac{1}{8}\log_2 8+\frac{1}{4}\log_2 4+\frac{5}{16}\log_2\frac{16}{5}+\frac{5}{16}\log_2\frac{16}{5}\\ =&1.924\end{aligned}$$

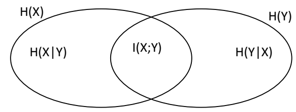
Y=length(X), i.e, Y is a deterministic function of X.

$$\begin{aligned}H(Y|X)=&0\\ H(X)+H(Y|X)=&H(X,Y)=H(Y)+H(X|Y)\\ \implies H(X|Y)=&H(X)-H(Y)\\ =&3.889\end{aligned}$$

5. a) $H(X)=\frac{2}{3}\log_2\frac{3}{2}+\frac{1}{3}\log_2 3=0.918=H(Y)$
   b) $H(X|Y)=\frac{1}{3}H(X|Y=0)+\frac{2}{3}H(X|Y=1)=0.667=H(Y|X)$
   c) $H(X,Y)=H(X)+H(Y|X)=1.585$
   d) $H(Y)-H(Y|X)=0.251$
   e) $I(X;Y)=H(Y)-H(Y|X)=0.251$
   f)

6. Identically distributed $\implies H(X_1)=H(X_2)$

   a) $\rho=1$ - $\frac{H(X_2|X_1)}{H(X_1)}=\frac{H(X_1)-H(X_2|X_1)}{H(X_1)}=\frac{H(X_2)-H(X_2|X_1)}{H(X_1)}=\frac{I(X_1;X_2)}{H(X_1)}$



b) $I(X_1;X_2) = H(X_1) - H(X_1|X_2)$
But $H(X_1|X_2) \geq 0 \implies I(X_1;X_2) \leq H(X_1) \implies \rho \leq 1$
$I(X_1;X_2) \geq 0 \implies \rho \geq 0$
$\therefore 0 \leq \rho \leq 1$

c) X_1,X_2 are independent.So $H(X_1|X_2) = H(X_1) \implies I(X_1;X_2) = 0$
Thus $\rho = 0$ when X_1 and X_2 are i.i.d

d) $H(X_2|X_1) = 0$ when X_2 is a function of $X_1 \implies \rho = 1$.