EE6340: Information Theory
Problem Set 3

1. *The AEP and source coding.* A discrete memoryless source emits a sequence of statistically independent binary digits with probabilities $p(1) = 0.005$ and $p(0) = 0.995$. The digits are taken 100 at a time and a binary codeword is provided for every sequences of 100 digits containing three or fewer ones.

   (a) Assuming that all codewords are the same length, find the minimum length required to provide codewords for all sequence with three or fewer ones.

   (b) Calculate the probability of observing a source sequence for which no codeword has been assigned.

2. X_1, X_2, $\cdots$, X_8 are i.i.d. binary random variables (i.e. $\mathcal{X} = \{0, 1\}$) with $Pr[X_1 = 1] = 7/8$.

   (a) For what values of ϵ does the typical set $A_\epsilon^{(8)}$ of sequences $(x_1, x_2, \cdots, x_8) \in \mathcal{X}^8$ have exactly 8 elements.

   (b) List the elements in the resulting typical set. What is the probability of this typical set?

   (c) Find the number of elements in the typical set as a function of ϵ.

3. *AEP.* Let X_i be iid $\sim p(x)$, $x \in \{1, 2, ..., m\}$. Let $\mu = E[X]$ and $H = -\sum p(x) \log p(x)$. Let $A^n = \{x^n \in \chi^n : | -\frac{1}{n} \log p(x^n) - H| \leq \epsilon\}$. Let $B^n = \{x^n \in \chi^n : |\frac{1}{n} \sum_{i=1}^n x_i - \mu| \leq \epsilon\}$.

   (a) Does $\Pr\{X^n \in A^n\} \to 1$?

   (b) Does $\Pr\{X^n \in A^n \cap B^n\} \to 1$?

   (c) Show that $|A^n \cap B^n| \leq 2^{n(H+\epsilon)}$ for all n.

   (d) Show that $|A^n \cap B^n| \geq \frac{1}{2} 2^{n(H-\epsilon)}$ for n sufficiently large.

4. *AEP-like limit.* Let $X_1, X_2, ...$ be i.i.d. drawn according to probability mass function $p(x)$. Find

$$\lim_{n \to \infty} (p(X_1, X_2, ...X_n))^{\frac{1}{n}}.$$

5. *AEP.* Let $X_1, X_2, ...$ be independent, identically distributed random variables drawn according to the probability mass function $p(x), x \in \{1, 2, ..., m\}$. Thus, $p(x_1, x_2, ...x_n) = \prod_{i=1}^n p(x_i)$. We know that $-\frac{1}{n} \log p(X_1, X_2, ...X_n) \to H(X)$ in probability. Let $q(x_1, x_2, ...x_n) = \prod_{i=1}^n q(x_i)$, where q is another probability mass function on $\{1, 2, 3.....m\}$.

   (a) Evaluate $\lim -\frac{1}{n} \log q(X_1, X_2, ...X_n)$, where $X_1, X_2, ...$ are i.i.d. $\sim p(x)$.

   (b) Now evaluate the limit of the log likelihood ratio $\frac{1}{n} log \frac{q(X_1, X_2, ...X_n)}{p(X_1, X_2, ...X_n)}$ when $X_1, X_2, ...$ are i.i.d.$\sim p(x)$. Thus, the odds favoring q are exponentially small when p is true.