EE613: Estimation Theory
Problem Set 10

1. If x and y are distributed according to a bivariate Gaussian PDF, then show that the conditional PDF $p(y|x)$ is also Gaussian with mean

$$E[y|x] = E[y] + \frac{\mathrm{cov}(x,y)}{\mathrm{var}(x)}(x - E[x])$$

and variance

$$\mathrm{var}(y|x) = \mathrm{var}(y) - \frac{\mathrm{cov}^2(x,y)}{\mathrm{var}(x)}.$$

2. The data $x[n]$ for $n = 0, 1, \ldots, N-1$ are observed, each sample having the conditional PDF

$$p(x[n]|\theta) = \begin{cases} \exp\left[-(x[n] - \theta)\right] & x[n] > \theta \\ 0 & x[n] < \theta \end{cases},$$

and conditioned on θ the observations are independent. The prior PDF is

$$p(\theta) = \begin{cases} \exp(-\theta) & \theta > 0 \\ 0 & \theta < 0 \end{cases}.$$

Find the MMSE estimator of θ.

3. Repeat the previous problem with the conditional PDF

$$p(x[n]|\theta) = \begin{cases} \dfrac{1}{\theta} & 0 \leq x[n] \leq \theta \\ 0 & \text{otherwise} \end{cases},$$

and the uniform prior PDF $\theta \sim \mathcal{U}[0, \beta]$. What happens if β is very large so that there is little prior knowledge?

4. In this problem we discuss reproducing PDFs. If $p(\theta)$ is chosen so that when multiplied by $p(\mathbf{x}|\theta)$ we obtain the same form of PDF in θ, then the posterior PDF $p(\theta|\mathbf{x})$ will have the same form as $p(\theta)$. Now assume that the PDF of $x[n]$ conditioned on θ is the exponential PDF

$$p(x[n]|\theta) = \begin{cases} \theta \exp(-\theta x[n]) & x[n] > 0 \\ 0 & x[n] < 0 \end{cases},$$

where the $x[n]$'s are conditionally independent. Next, assume the gamma prior PDF

$$p(\theta) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} \exp(-\lambda\theta) & \theta > 0 \\ 0 & \theta < 0 \end{cases}.$$

where $\lambda > 0$, $\alpha > 0$, and find the posterior PDF. Compare it to the prior PDF. Such a PDF, in this case the gamma, is termed a conjugate prior PDF.