EE611 Solutions to Problem Set 5

1. Let the coefficients of feed-forward filter be c_n, $n \in \{-3,-2,-1,0\}$ and of feedback filter be c_n, $n \in \{1,2\}$ and $f_n = 1, n \in \{0,1,2\}$. The feedback filter should be chosen as follows.

$$c_1 = -\left(0.25c_{-1} + c_0\right) \quad \text{and} \quad c_2 = -0.25c_0.$$

The feedforward filter coefficients can be determined using the orthogonality equations as follows.

$$\left(\begin{bmatrix} 1.125 & 0.5 & 0.0625 & 0 \\ 0.5 & 1.125 & 0.5 & 0.0625 \\ 0.0625 & 0.5 & 1.0625 & 0.25 \\ 0 & 0.0625 & 0.25 & 0.0625 \end{bmatrix} + \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}\right) \begin{bmatrix} c_{-3} \\ c_{-2} \\ c_{-1} \\ c_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.25 \\ 1 \\ 0.25 \end{bmatrix}$$

Therefore, we have $c_{-3} = 0.0287$, $c_{-2} = -0.1806$, $c_{-1} = 0.8826$, $c_0 = 0.2501$, $c_1 = -0.4708$, and $c_2 = -0.0625$.

2. (a) First we express $I(X;Y)$, the mutual information between the input and output of the Z-channel, as a function of $x = P[X = 1]$.

$$\begin{aligned} H(Y/X) &= P[x=0].0 + P[x=1].1 = x \\ H(Y) &= H(P[y=1]) = H(x/2) \\ I(X;Y) &= H(Y) - H(Y/X) - H(x/2) - x \end{aligned}$$

Since $I(X;Y) = 0$ when $x = 0$ and $x = 1$, the maximum mutual information is obtained for some value of x such that $0 < x < 1$.

$$\frac{d}{dx} I(X;Y) = \frac{1}{2}\log_2 \frac{1 - x/2}{x/2} - 1,$$

which is zero for $x = 2/5$. So, the capacity of the Z-channel in bits is $H(1/5) - 2/5 = 0.722 - 0.4 = 0.322$. It is reasonable that $x < 1/2$ because $X = 1$ is the noisy input to the channel.
(b) $I(X;Y) = H(Y) - H(Y/X) = H(Y) - H(p) \leq \log 3 - H(p)$. There is no input distribution $p_X(x)$ that gives $p_Y(y)$ to be uniform $\{0, e, 1\}$.
Let $P[X=0] = \alpha$, $P[Y=0] = \alpha(1-p)$, $P[Y=1] = (1-\alpha)(1-p)$, and $P[Y=e] = \alpha p + (1-\alpha)p$.

$$\begin{aligned} H(y) &= \alpha(1-p)\log\frac{1}{\alpha(1-p)} + (1-\alpha)(1-p)\log\frac{1}{(1-\alpha)(1-p)} + p\log\frac{1}{p} \\ &\quad p\log\frac{1}{p} + (1-p)\log\frac{1}{(1-p)} + (1-p)\left[\alpha\log\frac{1}{\alpha} + (1-\alpha)\log\frac{1}{(1-\alpha)}\right] \\ &= H(p) + (1-p)H(\alpha). \end{aligned}$$

$$I(X;Y) = (1-p)H(\alpha). \tag{1}$$

Therefore, by choosing the input to be uniform, we get the capacity $C = 1 - p$.
(c)$I(X;Y) = H(Y) - H(Y/X) = H(Y) - H(0.2)$. The capacity is achieved by making the output of the channel uniformly distributed. This can be achieved by choosing the input to be uniform. $C = 2 - H(0.2) = 2 - 0.72193 = 1.2781$ bits.

3. (a) At "low" SNR, the channel with higher SNR should be used. SNR of channel 1 is $|h_1|^2/N_1 = 2$ and SNR of channel 2 is $|h_2|^2/N_2 = 9/2$. Therefore, at "low" SNR, only channel 2 should be used.

   (b) If $P < N_2/|h_2|^2 - N_1/|h_1|^2 = 1/2 - 2/9 = 5/18$, only channel 2 is used. If $P > 5/18$, both channels will be used.

   (c) The water filling power allocation is as follows:

$$P_1 = (\nu - 1/2)^+ \qquad \text{(power allocated to channel 1)},$$

$$P_2 = (\nu - 2/9)^+ \qquad \text{(power allocated to channel 2)},$$

   and

$$P_1 + P_2 = P.$$

   The capacity using water-filling power allocation is

$$\frac{1}{2}\log_2\left(1 + \frac{P_1|h_1|^2}{N_1}\right) + \frac{1}{2}\log_2\left(1 + \frac{P_2|h_2|^2}{N_2}\right).$$

   Therefore, we have

$$C = \frac{1}{2}\log_2(1 + 2P)$$

   for $P < 5/18$ and

$$C = \frac{1}{2}\log_2(1 + 2P_1) + \frac{1}{2}\log_2(1 + 9P_2/2)$$

   for $P > 5/18$, where $P_1 = P/2 - 5/36$.
   The capacity using equal power allocation is

$$\frac{1}{2}\log_2\left(1 + \frac{P|h_1|^2}{2N_1}\right) + \frac{1}{2}\log_2\left(1 + \frac{P|h_2|^2}{2N_2}\right).$$

   See Figure 1 for the capacity plot.

4. The channel model is given by: $Y = X + Z$, where $X \in \{+1, -1\}$ and $Z \sim \mathcal{N}(0, \sigma^2)$. It has been shown that the mutual information $I(X;Y)$ subject to the constraint of BPSK signaling, is maximized for equiprobable signaling. As with the capacity without an input alphabet constraint, the capacity for BPSK also depends on these parameters only through their ratio, the SNR $1/\sigma^2$. Therefore, replace Y by Y/σ to get the model

$$Y = \sqrt{SNR}\, X + Z, \qquad X \in \{-1, +1\}, \qquad Z \sim \mathcal{N}(0, \sigma^2) \tag{2}$$

For notational simplicity, set $A = \sqrt{SNR}$, We have

$$p(y/+1) = \frac{1}{\sqrt{2\pi}} e^{-(y-A)^2/2}$$

$$p(y/-1) = \frac{1}{\sqrt{2\pi}} e^{-(y+A)^2/2}$$

and

$$p(y) = \frac{1}{2} p(y/+1) + \frac{1}{2} p(y/-1) \tag{3}$$

We can compute

$$I(X;Y) = h(Y) - h(Y/X)$$

$h(Y/X) = h(Z) = 1/2 \log_2(2\pi e)$, since $Z \sim \mathcal{N}(0, \sigma^2)$. $h(y) = -\int \log_2(p_Y(y)) p_Y(y) dy$ can be computed by numerical integration, plugging in eq. (3). An alternative approach, which is particularly useful for more complicated constellations and channel models, is to use Monte carlo integration (i.e., simulation-based empirical averaging) for computing the expectation $h(Y) = -E[log_2(p(Y)]$. For this method, we generate i.i.d. samples $Y_1, \cdots Y_n$ and then use the estimate

$$\hat{h}(Y) = \frac{1}{n} \sum_{i=1}^{n} log_2 p(Y_i). \tag{4}$$

We can also use the alternative formula

$$I(X;Y) = H(X) - H(X/Y) \tag{5}$$

to compute the capacity. For equiprobable binary input, $H(X) = H_B\left(\frac{1}{2}\right) = 1$ bits/symbol. It remains to compute

$$H(X/Y) = \int H(X/Y=y) p_Y(y) dy \tag{6}$$

By Baye's rule, we have

$$\begin{aligned} P[X=+1/Y=y] &= \frac{P[X=+1]p(y/+1)}{p(y)} \\ &= \frac{P[X=+1]p(y/+1)}{P[X=+1]p(y/+1) + P[X=-1]p(y/-1)} \\ &= \frac{e^{Ay}}{e^{Ay}+e^{-Ay}} \text{(equal priors)} \end{aligned} \tag{7}$$

We also have

$$P[x=+1]Y=y] = 1 - P[X=+1/Y=y] = \frac{e^{-Ay}}{e^{Ay}+e^{-Ay}} \tag{8}$$

Such a posteriori probability computations can be thought of as *soft decisions* on the transmitted bits, and are employed extensively when we discuss iterative decoding. We can now use the binary entropy function to compute

$$H(X/Y=y) = H_B(P[X=+1/Y=y]) \tag{9}$$

The average in eq. (6) can now be computed by either direct numerical integration or by Monte carlo integration as before. The later, which generalizes better to more complex models, gives the estimate

$$\hat{H}(X/Y) = \frac{1}{n}\sum_{i=1}^{n} H_B(P[X = +1/Y_i = y_i]) \tag{10}$$

The plot of the mutual information of BPSK signaling in AWGN channel is shown in the following figure

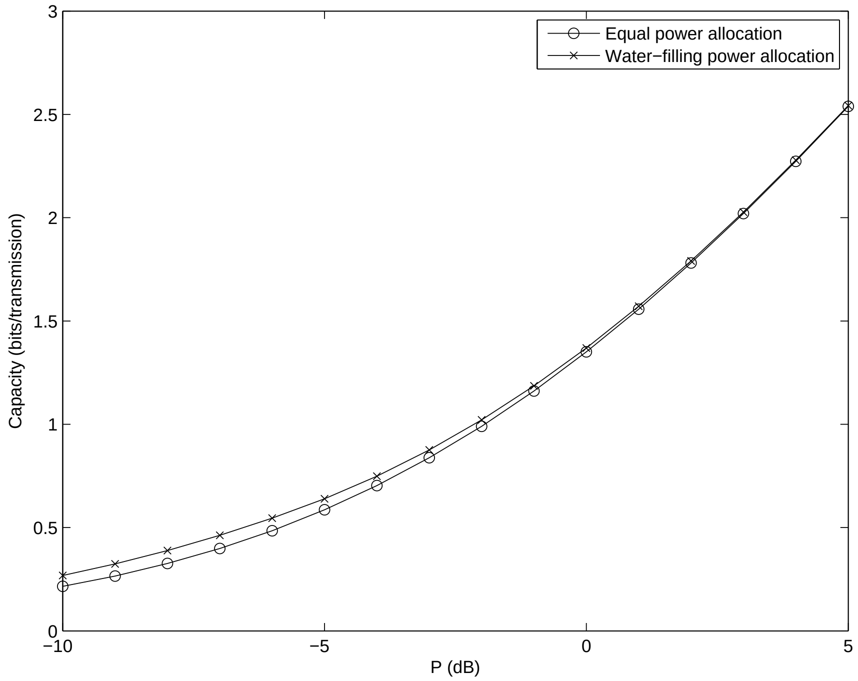

Figure 1: Comparison of capacities with equal power allocation and water-filling power allocation

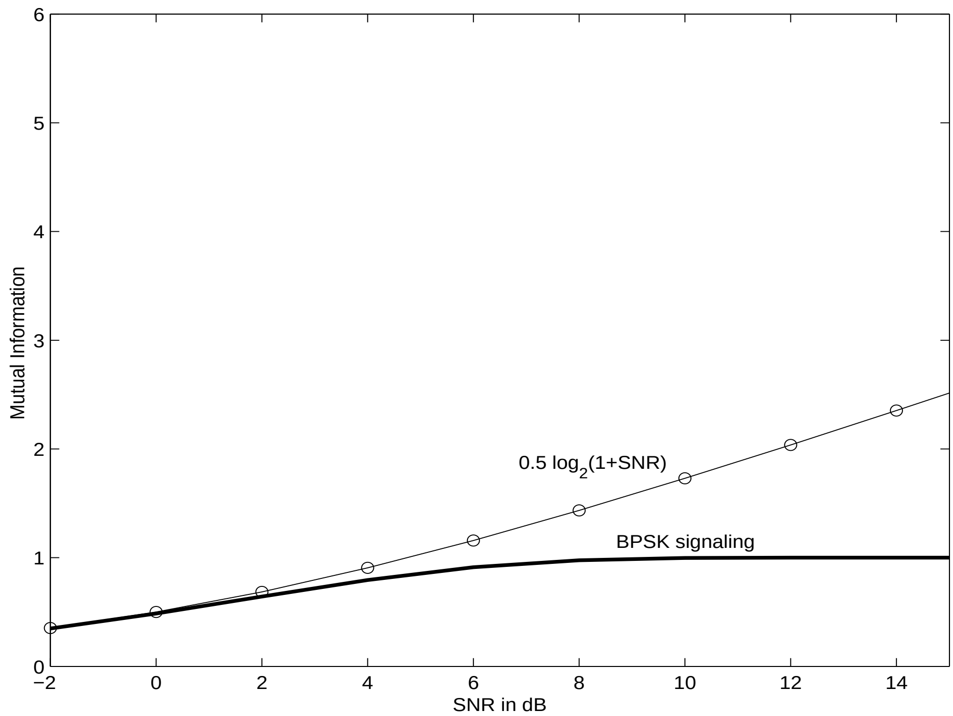

Figure 2: Mutual Information for BSPK signaling vs. SNR