EE611 Solutions to Problem Set 3

1. (a) We have

$$s_i(t) = \sqrt{\frac{2E_s}{T}} \cos \frac{2\pi kt}{T} \cos \frac{2\pi i}{M} - \sqrt{\frac{2E_s}{T}} \sin \frac{2\pi kt}{T} \sin \frac{2\pi i}{M}.$$

Using $f_1(t) = \sqrt{\frac{2}{T}} \cos \frac{2\pi kt}{T}$ and $f_2(t) = -\sqrt{\frac{2}{T}} \sin \frac{2\pi kt}{T}$ as the two orthonormal basis functions, we get

$$\underline{s}_i = \left(\sqrt{E_s} \cos \frac{2\pi i}{M} \quad \sqrt{E_s} \sin \frac{2\pi i}{M} \right).$$

Since the signals are equally likely, the optimum decision regions are obtained by using the minimum distance criterion. The signal constellation and optimum decision regions for $M = 5$ are shown in Figure 1.

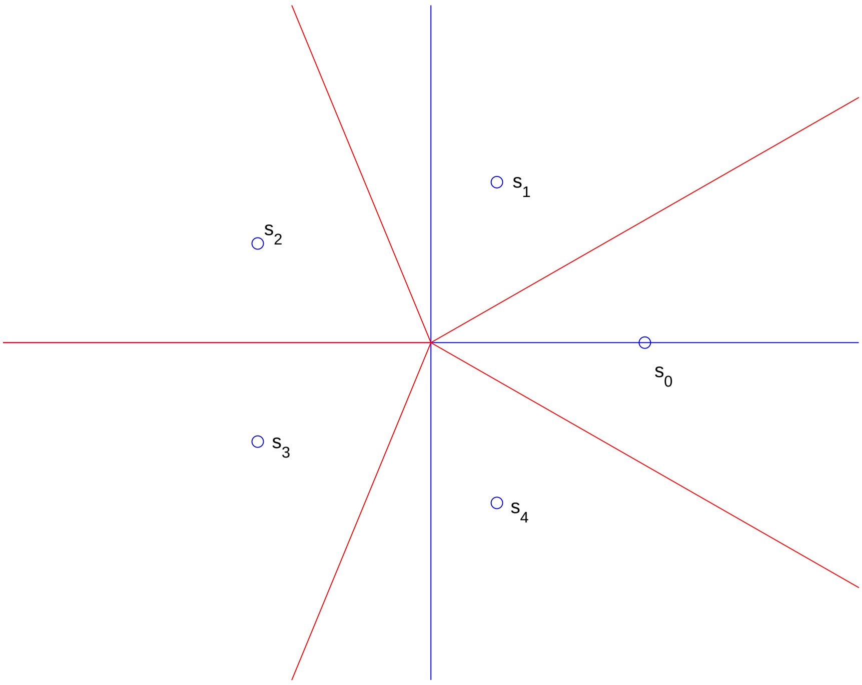


Figure 1: Signal Constellation and Decision Regions: Decision boundaries are shown in red.

(b) By symmetry, we have $P[\varepsilon] = P[\varepsilon|\underline{s}_0] = P[\underline{r} \in \bar{R}_0|\underline{s}_0]$, where $\bar{R}_0$ is the complement of R_0. Now, consider the two regions D_1 and D_2 shown Figure 2. We have

$$D_1 \subset \bar{R}_0 = D_1 \cup D_2$$

$$\Rightarrow P[\underline{r} \in D_1|\underline{s}_0] \leq P[\varepsilon] = P[\underline{r} \in D_1 \cup D_2|\underline{s}_0] \leq P[\underline{r} \in D_1|\underline{s}_0] + P[\underline{r} \in D_2|\underline{s}_0].$$

Since

$$P[\underline{r} \in D_1|\underline{s}_0] = P[\underline{r} \in D_2|\underline{s}_0] = p = Q\left(\frac{d}{2\sigma}\right),$$

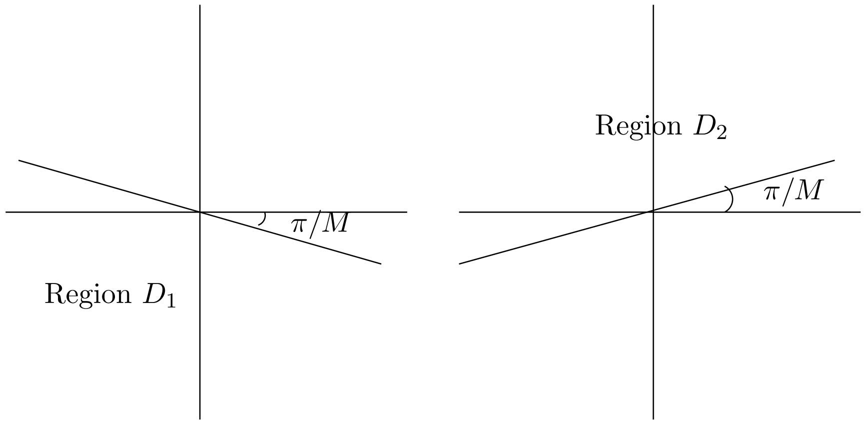


Figure 2:

where $d = 2\sqrt{E_s}\sin(\pi/M)$ and $\sigma = \sqrt{N_0/2}$, we get

$$p \leq P[\varepsilon] \leq 2p,$$

where

$$p = Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right).$$

2. The optimum decision regions are shown in Figure 3.

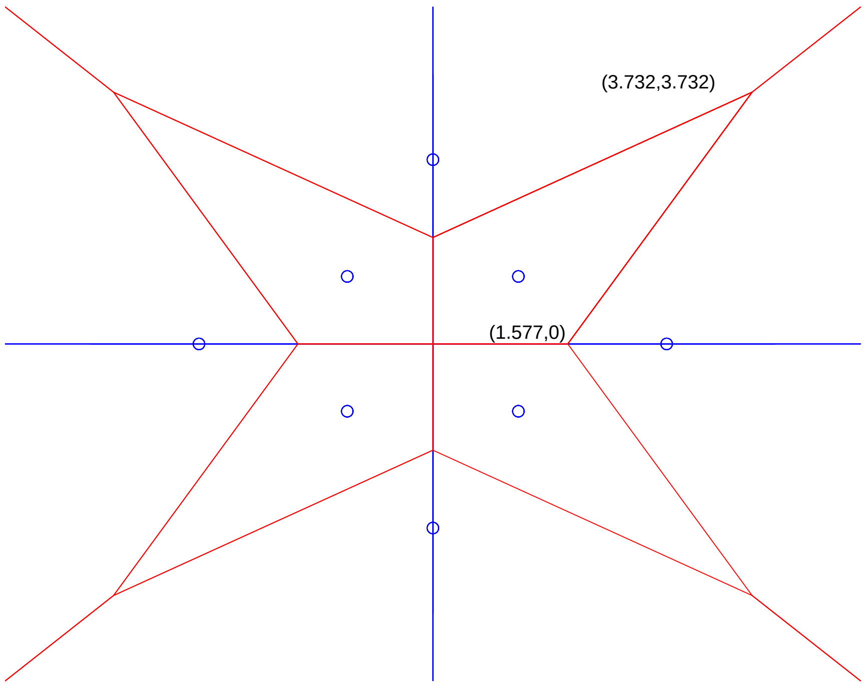


Figure 3: Decision Regions: Decision Boundaries are shown in red.

3. (a) The equivalent low pass channel has bandwidth $W = 1350$ Hz. Therefore, we have $2W = 2700$ Hz. For zero ISI, we need

$$\frac{1}{T} < 2W.$$

The required bit rate is 9600 bits per second. If the number of bits per symbol is k, we have

$$\frac{1}{T} = \frac{9600}{k} < 2700.$$

Therefore

$$k > \frac{96}{27}.$$

We need to choose k such that (i) it is an integer, and (ii) it is as small as possible because power efficiency of QAM decreases as k increases. Therefore, $k = 4$ corresponding to a symbol rate of 2400 symbols per second is the best possible choice. For $k = 4$, a 16-QAM constellation is used.

(b) The bandwidth of a square-root raised cosine pulse with roll-off factor β is $(1+\beta)/2T$. Therefore, we have

$$\frac{1+\beta}{T} = 2700 \Rightarrow 1+\beta = \frac{9}{8} \Rightarrow \beta = 0.125.$$

4. (a) The raised cosine spectrum is real and given by

$$X(f) = \begin{cases} T & |f| < \frac{1-\beta}{2T} \\ \frac{T}{2}\left[1 + \cos\left\{\frac{\pi T}{\beta}\left(|f| - \frac{1-\beta}{2T}\right)\right\}\right] & \frac{1-\beta}{2T} \le |f| \le \frac{1+\beta}{2T} \\ 0 & |f| > \frac{1+\beta}{2T} \end{cases}$$

We want to express $X(f)$ in the range $0 \le f \le \frac{1}{T}$ as $G_1(f) + G_2(f)$, where $G_1(f)$ is a rectangular function and $G_2(f)$ is an odd function around $\frac{1}{2T}$. It can be done in the following manner (See Figure 4):

$$G_1(f) = \frac{T}{2} \quad \text{for } 0 \le f \le \frac{1}{T},$$

and

$$G_2(f) = \begin{cases} \frac{T}{2} & 0 \le f \le \frac{1-\beta}{2T} \\ \frac{T}{2}\cos\left\{\frac{\pi T}{\beta}\left(f - \frac{1-\beta}{2T}\right)\right\} & \frac{1-\beta}{2T} < f < \frac{1+\beta}{2T} \\ -\frac{T}{2} & \frac{1+\beta}{2T} \le f \le \frac{1}{T} \end{cases}$$

(b) Choose $G_1(f)$ as in part (a). Choose $G_2(f)$ as follows:

$$G_2(f) = \begin{cases} \frac{T}{2} & 0 \le f \le \frac{1-\beta}{2T} \\ T\left\{\frac{T}{\beta}\left(-f + \frac{1}{2T}\right)\right\} & \frac{1-\beta}{2T} < f < \frac{1+\beta}{2T} \\ -\frac{T}{2} & \frac{1+\beta}{2T} \le f \le \frac{1}{T} \end{cases}$$

5. The Fourier transform of $x(t)$ can be determined as

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} e^{-\pi a^2 t^2} e^{-j2\pi ft} dt \\ &= e^{-\frac{\pi f^2}{a^2}} \int_{-\infty}^{\infty} e^{-\pi a^2 (t + j\frac{f}{a^2})^2} dt \\ &= \frac{1}{a} e^{-\frac{\pi f^2}{a^2}} \end{aligned}$$

Setting $x(T) = 0.01$ gives $aT = \sqrt{\frac{2 \ln 10}{\pi}}$.
$\frac{X(W)}{X(0)} = 0.01$ implies $W = \frac{2 \ln 10}{\pi T} = 1.46587 \frac{1}{T}$.
Raised-cosine spectrum has bandwidth $W = \frac{1+\beta}{2T} = \frac{1}{T}$ for $\beta = 1$.

6. Given the expression for $x_{rc}(t)$, it can be easily seen that $x(0) = 1$. Therefore, $\int_{-\infty}^{\infty} X_{rc}(f) df = 1$. Given the raised cosine spectrum expression $X_{rc}(f)$ in frequency domain as follows, we can also evaluate the integral easily and show that it is 1.

$$X_{rc}(f) = \begin{cases} T & 0 \le |f| \le \frac{1-\beta}{2T} \\ \frac{T}{2}\left\{1 + cos\left[\frac{\pi T}{\beta}\left(|f| - \frac{(1-\beta)}{2T}\right)\right]\right\} & \frac{1-\beta}{2T} \le |f| \le \frac{1+\beta}{2T} \\ 0 & |f| > \frac{1+\beta}{2T} \end{cases} .$$

$$\int_{-\infty}^{\infty} X_{rc}(f) df = \int_{-\frac{(1+\beta)}{2T}}^{-\frac{(1-\beta)}{2T}} X_{rc}(f) df + \int_{-\frac{(1-\beta)}{2T}}^{\frac{(1-\beta)}{2T}} X_{rc}(f) df + \int_{\frac{1-\beta}{2T}}^{\frac{1+\beta}{2T}} X_{rc}(f) df$$

- Terms 1 and 3 on the right side are equal and can be evaluated as follows.

$$\begin{aligned} \int_{\frac{1-\beta}{2T}}^{\frac{1+\beta}{2T}} X_{rc}(f) df &= \int_{\frac{(1-\beta)}{2T}}^{\frac{(1+\beta)}{2T}} \frac{T}{2}\left\{1 + cos\left[\frac{\pi T}{\beta}\left(f - \frac{1-\beta}{2T}\right)\right]\right\} df \\ \int_{-\frac{(1+\beta)}{2T}}^{-\frac{(1-\beta)}{2T}} X_{rc}(f) df &= \int_{-\frac{(1+\beta)}{2T}}^{-\frac{(1-\beta)}{2T}} \frac{T}{2}\left\{1 + cos\left[\frac{\pi T}{\beta}\left(-f - \frac{(1-\beta)}{2T}\right)\right]\right\} df \\ (\text{ by setting } \alpha = -f) &= \int_{\frac{(1-\beta)}{2T}}^{\frac{(1+\beta)}{2T}} \frac{T}{2}\left\{1 + cos\left[\frac{\pi T}{\beta}\left(\alpha - \frac{(1-\beta)}{2T}\right)\right]\right\} d\alpha \\ &= \int_{\frac{1-\beta}{2T}}^{\frac{1+\beta}{2T}} X_{rc}(f) df \end{aligned}$$

$$\begin{aligned} 2\int_{\frac{1-\beta}{2T}}^{\frac{1+\beta}{2T}} X_{rc}(f) df &= \int_{\frac{(1-\beta)}{2T}}^{\frac{(1+\beta)}{2T}} \left\{T + T cos\left[\frac{\pi T}{\beta}\left(f - \frac{1-\beta}{2T}\right)\right]\right\} df \\ &= T\left(\frac{(1+\beta)}{2T} - \frac{(1-\beta)}{2T}\right) + T\int_{\frac{1-\beta}{2T}}^{\frac{1+\beta}{2T}} cos\left(\frac{\pi T f}{\beta} - \frac{(1-\beta)\pi}{2\beta}\right) df \end{aligned}$$

$$\left(\text{setting } \alpha = \frac{\pi T}{\beta}\left(f - \frac{(1-\beta)}{2T}\right), \text{ we get limits } \alpha = 0 \text{ to } \pi\right)$$

$$\begin{aligned} &= \beta + T\int_0^{\pi} cos\alpha\ d\alpha\left(\frac{\beta}{\pi T}\right) \\ &= \beta + \frac{\beta}{\pi}\int_0^{\pi} cos\alpha\ d\alpha = \beta + 0 = \beta \end{aligned}$$

- Term 2 is evaluated as follows.

$$\int_{\frac{-(1-\beta)}{2T}}^{\frac{(1-\beta)}{2T}} X_{rc}(f)df = T\left(\frac{1-\beta}{2T}\right)2 = 1-\beta.$$

- Therefore, we have

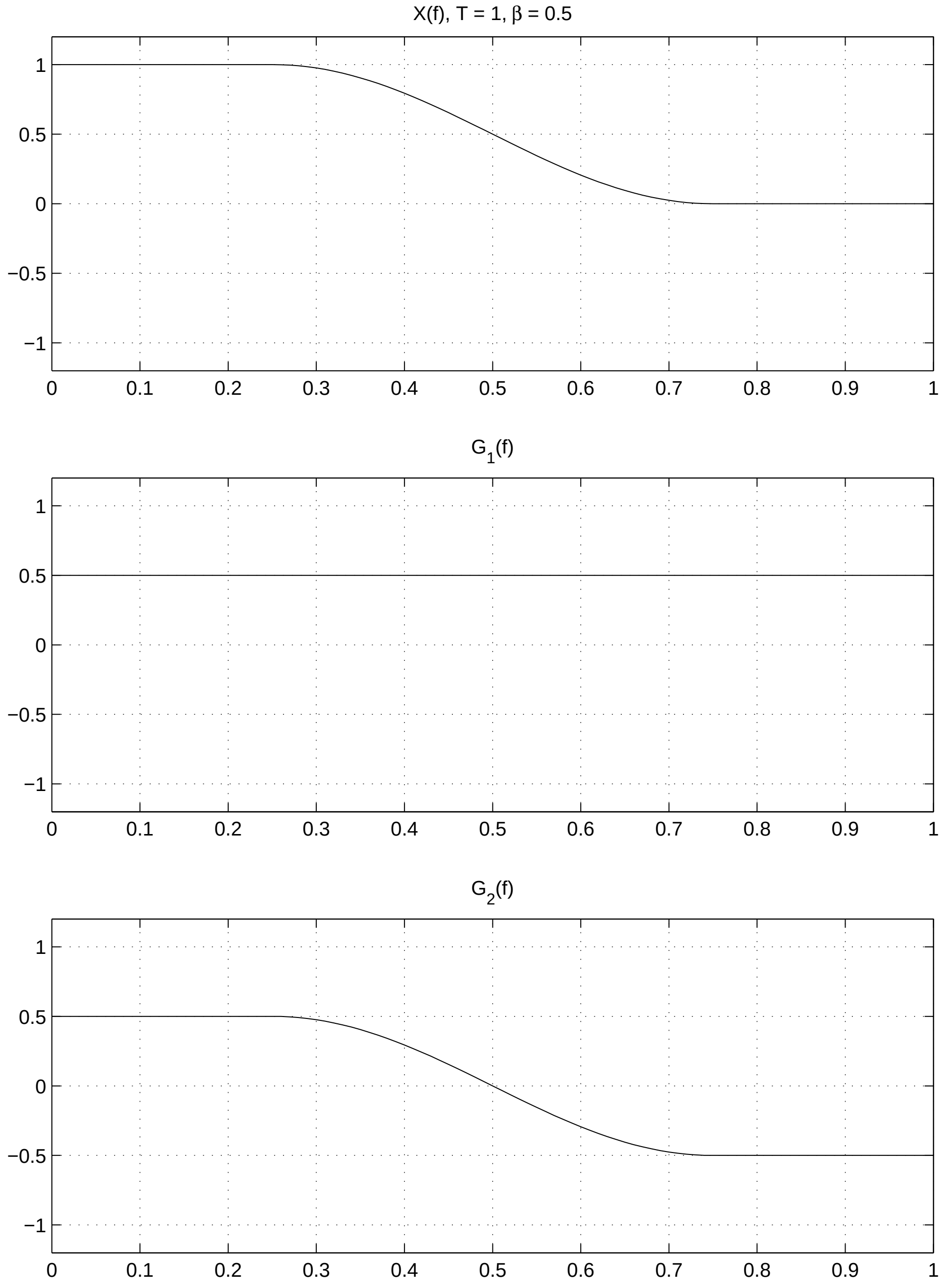
$$\int_{-\infty}^{\infty} X_{rc}(f)df = 1-\beta+\beta = 1.$$

7. Assuming $X_{rc}(f)$ to be defined as in the previous problem, the variance of the filtered noise will be
$$\frac{N_0}{2}\int_{-\infty}^{\infty} X_{rc}(f)df = \frac{N_0}{2}.$$

8. Bandwidth of raised cosine spectrum with roll-off factor of β is $\frac{1}{2T}(1+\beta)$.

   For zero ISI, we want $\frac{1}{T}(1+\beta) \leq 2W \Rightarrow \frac{1}{T} \leq \frac{2W}{1+\beta}$ where $2W = 4kHz$

   Since $\beta \geq 0.5$, we have $\frac{1}{T} \leq \frac{2W}{1.5} = \frac{8}{3}kHz = 2.667kHz$
   Since the required bit rate = 9600 bps, and we want to choose as few bits per symbol as possible, we can select 16-QAM at a symbol rate of $\frac{9600}{4} = 2400$ symbols per second $<$ 2.667 ksps.



X(f), T = 1, β = 0.5
1
0.5
0
−0.5
−1
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
G1(f)
1
0.5
0
−0.5
−1
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
G2(f)
1
0.5
0
−0.5
−1
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

Figure 4: