EE5160: Error Control Coding
Problem Set 7

1. Find the power series $s(D)$ for the rational function

$$\frac{a(D)}{b(D)} = \frac{1+D+D^2}{1+D+D^3}$$

and note that, after a transient, its coefficients are periodic. Show that $a(D) = b(D)s(D)$.

2. Find $G_{sys}(D), H_{sys}(D), G_{poly}(D)$ and $H_{poly}(D)$ for the rate-2/3 convolutional-code generator matrix given by

$$G(D) = \begin{bmatrix} \frac{1+D}{1+D+D^2} & 1+D & 0 \\ 1 & \frac{1}{1+D} & \frac{D}{1+D^2} \end{bmatrix}.$$

3. Find the Type I realization of the transfer function

$$g_{ij}(D) = \frac{a_0 + a_1 D + ... + a_m D^m}{b_0 + b_1 D + ... + b_m D^m}.$$

To do so write $c^{(j)}(D)$ as

$$c^{(j)}(D) = \left(u^{(i)}(D) \cdot \frac{1}{b(D)}\right) \cdot a(D).$$

Then sketch the direct implementation of the leftmost "filtering" operation $v(D) = u^{(i)}(D) \cdot 1/b(D)$, which can be determined from the difference equation

$$v_t = u_t^{(i)} - b_1 v_{t-1} - b_2 v_{t-2} - \cdots - b_m v_{t-m}.$$

Next, sketch the direct implementation of the second filtering operation $c^{(j)}(D) = v(D)a(D)$, which can be determined from the difference equation

$$c_t^{(j)} = a_0 v_t^{(i)} + a_1 v_{t-1}^{(i)} \cdots + a_m v_{t-m}^{(i)}.$$

Finally, sketch the two "filters" in cascade, the $1/b(D)$ filter followed by the $a(D)$ filter (going from left to right), and notice that the m delay (memory) elements may be shared by the two filters.

4. Consider the rate-2/3 convolutional code with

$$G(D) = \begin{bmatrix} \frac{1+D}{1+D+D^2} & 0 & 1+D \\ 1 & \frac{1}{1+D} & \frac{D}{1+D^2} \end{bmatrix}.$$

Let the input for the encoder G(D) be $u(D) = \begin{bmatrix} 1 & 1+D \end{bmatrix}$ and find the corresponding codeword $c(D) = \begin{bmatrix} c_1(D) & c_2(D) & c_3(D) \end{bmatrix}$. Find the input which yields the same codeword when the encoder is given by $G_{sys}(D)$. Repeat for $G_{poly}(D)$.

5. Consider a rate-2/3 convolutional code with

$$G_{poly}(D) = \begin{bmatrix} 1+D & 0 & 1 \\ 1+D^2 & 1+D & 1+D+D^2 \end{bmatrix}.$$

Show that the memory required for the Type I and TypeII realizations of $G_{poly}(D)$ is $\mu = 3$ and $\mu = 6$, respectively. Show that

$$G_{sys}(D) = \begin{bmatrix} 1 & 0 & \frac{1}{1+D} \\ 0 & 1 & \frac{D^2}{1+D} \end{bmatrix},$$

and that the memory required for its Type I realization is $\mu = 3$. Finally, show that the Type II realization of $G_{sys}(D)$ requires only$\mu = 2$. Thus, the Type II realization of$G_{sys}(D)$ is the minimal encoder for rate $k/k+1$ convolutional codes.

6. Show that

$$\max(x, y) = \log\left(\frac{e^x + e^y}{1 + e^{-|x-y|}}\right).$$

Hint: First suppose $x > y$.

7. Draw the state diagram for the rate-1/2 encoder described by

$$u(D) = \begin{bmatrix} 1+D+D^2 & 1+D^2 \end{bmatrix}.$$

Find the input output weight enumerator (IO-WE)

$$A'(I, W) = \sum_{i,w} A'_{i,w} I^i W^w$$

where $A'_{i,w}$ is the number of weight-w paths, corresponding to weight-i encoder inputs, that diverge from the all-zeros trellis path one time in L trellis stages.

8. **(Optional)** Assuming the BI-AWGNC, simulate Viterbi decoding of the rate-1/2 convolutional code whose encoder matrix is given by

$$G(D) = \begin{bmatrix} 1+D^2+D^3+D^4 & 1+D+D^4 \end{bmatrix}.$$

Plot the bit error rate from $P_b = 10^{-1}$ to $P_b = 10^{-6}$. Repeat for the BCJR decoder and comment on the performance of the two decoders.