# EE5130: Detection and Estimation Theory
## Problem Set 4

1. Consider the following Bayes decision problem. The conditional density of the real observation $Y$ given the real parameter $\Theta = \theta$ is given by

$$p_\theta(y) = \begin{cases} \theta e^{-\theta y}, & y \geq 0 \\ 0, & y < 0 \end{cases}.$$

$\theta$ is a random variable with density

$$\omega(\theta) = \begin{cases} \alpha e^{-\alpha \theta}, & \theta \geq 0 \\ 0, & \theta < 0 \end{cases}.$$

where $\alpha > 0$. Find the Bayes rule and minimum Bayes risk for the hypothesis

$$H_0 : \Theta \in (0, \beta) \triangleq \Lambda_0$$
$$\text{versus}$$
$$H_1 : \Theta \in (\beta, \infty) \triangleq \Lambda_1$$

where $\beta > 0$ is fixed. Assume the cost structure

$$C[i, \theta] = \begin{cases} 1, & \theta \ni \Lambda_i \\ 0, & \theta \in \Lambda_i \end{cases}.$$

2. Consider the hypothesis testing problem

$$H_0 : Y \text{ has density } p_0(y) = \frac{1}{2} e^{-|y|}, y \in R$$
$$\text{versus}$$
$$H_1 : Y \text{ has density } p_\theta(y) = \frac{1}{2} e^{-|y-\theta|}, y \in R, \theta > 0$$

   1. Describe the locally most powerful $\alpha$ level test and derive its power function.
   2. Does a uniformly most powerful test exist? If so, find it and derive its power function. If not, find the generalized likelihood ratio test for $H_0 versus H_1$.

3. Suppose the random observation vector $\underline{Y}$ is given by

$$Y_k = N_k + \theta S_k, k = 1, 2....n$$

where $\underline{N}$ is a zero mean Gaussian random vector with $E(N_k N_l) = \sigma^2 \rho^{|k-l|}$ for all $0 \leq k, l \leq n, |\rho| < 1$ and where $\underline{s}$ is a known signal vector.

a) Show that the test

$$\delta(\underline{y}) = \begin{cases} 1, & \Sigma_{k=1}^{n} b_k z_k \geq \tau' \\ 0, & \Sigma_{k=1}^{n} b_k z_k < \tau' \end{cases}.$$

is equivalent to the likelihood ratio test for $\theta = 0$ versus $\theta = 1$, where

$$b_1 = s_1/\sigma$$
$$b_k = (s_k - \rho s_{k-1})/\sigma\sqrt{1-\rho^2}, k = 2, 3, ...n$$
$$z_1 = y_1/\sigma$$
$$z_k = (y_k - \rho y_{k-1})/\sigma\sqrt{1-\rho^2}, k = 2, 3, ...n$$

b) Find the ROCs of the detector from (a) as a function of $\theta/\sigma, \rho, n$, and the false alarm probability $\alpha$.

4. Consider the M-ary decision problem ($\Gamma = R^n$)

$$H_0 : \underline{Y} = \underline{N} + \underline{s_0}$$
$$H_1 : \underline{Y} = \underline{N} + \underline{s_1}$$
$$\vdots$$
$$H_{M-1} : \underline{Y} = \underline{N} + \underline{s_{M-1}}$$

where $\underline{s_0}, \underline{s_1}, ....., \underline{s_{M-1}}$ are known signals with equal energies $||s_0||^2 = ||s_1||^2 = .... = ||s_{M-1}||^2$.

a) Assuming $\underline{N} = N(\underline{0}, \sigma^2\mathbf{I})$, find the decision rule achieving minimum error probability when all hypothesis are equally likely.

b) Assuming further that the signals are orthogonal, show that the minimum error probability is given by

$$P_e = 1 - \frac{1}{2\pi} \int_{\infty}^{\infty} [\phi(x)]^{M-1} e^{\frac{-(x-d)^2}{2}} dx$$

where $d^2 = ||s_0||^2/\sigma^2$

5. Consider the following hypothesis about a sequence $Y_1, Y_2, ...., Y_n$ of real observations.

$$H_0 : Y_k = N_k - s_k k = 1, 2, ....n$$
$$H_1 : Y_k = N_k k = 1, 2, ...., n$$
and
$$H_2 : Y_k = N_k + s_k k = 1, 2, ...., n$$

where $s_1, s_2, ..., s_N$ is a known signal sequence and $N_1, N_2, ..., N_n$ is a sequence of iid $N(0, 1)$ random variables.

a) Assuming that these three hypothesis are equally likely, find the decision rule minimizing the average probability of error in deciding among the three hypothesis.

b) Again assuming equally likely hypothesis, calculate the minimum average error probability for deciding among these hypotheses.