EE 511 Solutions to Problem Set 3

1. (a) $F_Y(y|X = 1/4) = P[Y \leq y|X = 1/4] = P[X + N \leq y|X = 1/4] = P[1/4 + N \leq y|X = 1/4] = P[N \leq y - 1/4|X = 1/4]$. Since X and N are independent, we have $P[N \leq y - 1/4|X = 1/4] = P[N \leq y - 1/4]$ and

$$F_Y(y|X = 1/4) = F_N(y - 1/4).$$

Therefore,

$$f_Y(y|X = 1/4) = f_N(y - 1/4)$$

i.e., uniform over (-1/4,3/4).

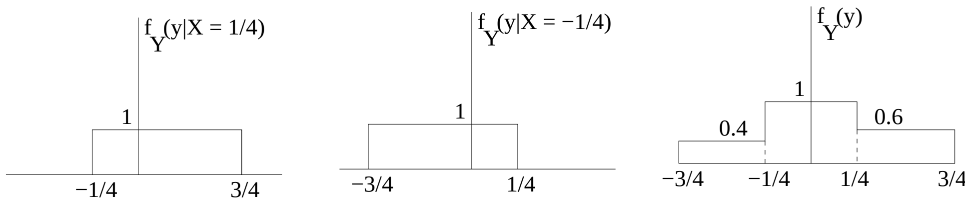
Similarly, we get

$$f_Y(y|X = -1/4) = f_N(y + 1/4)$$

i.e., uniform over (-3/4,1/4).

Now,

$$f_Y(y) = P[X = 1/4]f_Y(y|X = 1/4) + P[X = -1/4]f_Y(y|X = -1/4)$$



(b) Let C denote the event that we make a correct decision. We have

$$P[C] = \int_{-\infty}^{\infty} P[C|Y = y]f_Y(y)dy$$

Since the integrand in the above equation is always positive, maximizing $P[C]$ is the same as maximizing $P[C|Y = y]$ for each y. Therefore, the optimal rule is to choose

$$\text{Decision } = 1/4 \text{ if } P[X = 1/4|Y = y] > P[X = -1/4|Y = y]$$

$$\text{Decision } = -1/4 \text{ if } P[X = -1/4|Y = y] > P[X = 1/4|Y = y]$$

This can also be written as

$$\text{Decision } = 1/4 \text{ if } \frac{f_Y[y|X = 1/4]P[X = 1/4]}{f_Y(y)} > \frac{f_Y[y|X = -1/4]P[X = -1/4]}{f_Y(y)}$$

$$\text{Decision } = -1/4 \text{ if } \frac{f_Y[y|X = -1/4]P[X = -1/4]}{f_Y(y)} > \frac{f_Y[y|X = 1/4]P[X = 1/4]}{f_Y(y)}$$

i.e.,

$$\text{Decision } = 1/4 \text{ if } f_Y[y|X = 1/4]P[X = 1/4] > f_Y[y|X = -1/4]P[X = -1/4]$$

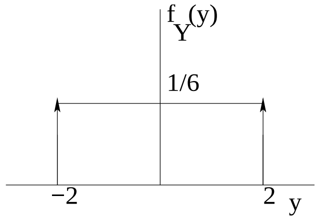
$$\text{Decision } = -1/4 \text{ if } f_Y[y|X = -1/4]P[X = -1/4] > f_Y[y|X = 1/4]P[X = 1/4]$$

In this case, we have Decision= 1/4 if $y \geq -1/4$ and Decision $= -1/4$ if $y < -1/4$.

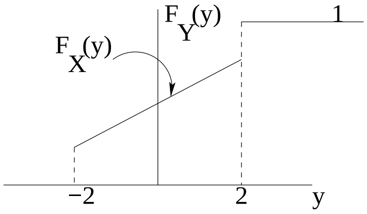
2. We have $Y = g(X)$. $F_Y(g(\alpha)) = P[Y \le g(\alpha)]$. Since $g(x)$ is a monotonically increasing function in x, it has an inverse and $Y \le g(\alpha)$ is equivalently $g^{-1}(Y) \le \alpha$. Therefore, we have

$$F_Y(g(\alpha)) = P[Y \le g(\alpha)] = P[g^{-1}(Y) \le \alpha] = P[X \le \alpha] = F_X(\alpha).$$

3. For $-2 \le y < 2$, $F_Y(y) = P[Y \le y] = P[X \le y] = F_X(y)$. For $y < -2$, $F_Y(y) = 0$ since $Y \ge -2$. Similarly, for $y \ge 2$, $F_Y(y) = 1$ since $Y \le 2$. Therefore, the cdf is as shown below.



Uniform pdf with 2 delta functions at −2 and 2



Using the cdf, we get the pdf to be

$$f_Y(y) = P[X \le -2]\delta(y+2) + f_X(y) + P[X > 2]\delta(y-2)$$

for $-2 \le y \le 2$ and $f_Y(y) = 0$ otherwise.

4. $Y = X^2$. For $y \ge 0$, $F_Y(y) = P[Y \le y] = P[X^2 \le y] = P[-\sqrt{y} \le X \le \sqrt{(y)}] = F_X(\sqrt{y}) - F_X(-\sqrt{y})$. Therefore, we have

$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$

(a)

$$f_Y(y) = \frac{1}{2\alpha} \exp\left\{-\frac{y}{2\alpha}\right\}$$

(b)

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma\sqrt{y}} \exp\left\{-\frac{y}{2\sigma^2}\right\}$$

5. For $y \ge 0$, we have

$$F_Y(y|X>0) = P[Y \le y|X>0] = P[X^2 \le y|X>0] = \frac{P[X^2 \le y, X > 0]}{P[X>0]} = \frac{P[0 < X \le \sqrt{y}]}{P[X>0]}$$

Therefore, we have

$$F_Y(y|X>0) = \frac{F_X(\sqrt{y}) - F_X(0)}{1 - F_X(0)}$$

From this, we get

$$f_Y(y|X>0) = \frac{f_X(\sqrt{y})}{2\sqrt{y}(1 - F_X(0))}$$

for $y \ge 0$.

6. $g(.) = F_Y^{-1}(.)$ or $Y = F_Y^{-1}(X)$ is the solution. First, let us determine $F_Y(y)$. For $y \geq 0$, we have

$$F_Y(y) = 1 - \int_y^{\infty} \frac{e^{-\sqrt{2}y}}{\sqrt{2}} dy = \frac{1}{2} e^{-\sqrt{2}y}\Big|_y^{\infty} + 1 = 1 - \frac{1}{2} e^{-\sqrt{2}y}$$

For $y < 0$, we have

$$F_Y(y) = \int_{-\infty}^{y} \frac{e^{\sqrt{2}y}}{\sqrt{2}} dy = \frac{e^{\sqrt{2}y}}{2}\Big|_{-\infty}^{y} = \frac{e^{\sqrt{2}y}}{2}$$

Therefore, we get $g(x)$ to be

$$g(x) = F_Y^{-1}(x) = \begin{cases} \frac{\log(2x)}{\sqrt{2}} & 0 \leq x < 0.5 \\ -\frac{\log(2-2x)}{\sqrt{2}} & 0.5 \leq x \leq 1 \end{cases}$$

7. $Z = X + Y$ where X and Y are independent random variables. Therefore, the pdf of Z is the convolution of the pdf's of X and Y.

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(\alpha) f_Y(z-\alpha) d\alpha$$

For $z < 0$ and $z > 3$, $f_Z(z) = 0$. For $0 \leq z \leq 1$, we have

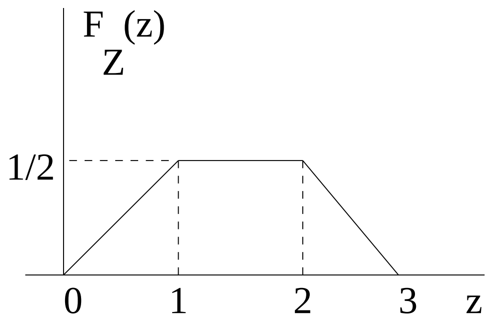
$$f_Z(z) = \int_0^z \frac{1}{2} d\alpha = \frac{z}{2}.$$

For $1 < z \leq 2$, we have

$$f_Z(z) = \int_0^1 \frac{1}{2} d\alpha = \frac{1}{2}.$$

For $2 < z \leq 3$, we have

$$f_Z(z) = \int_{z-2}^{1} \frac{1}{2} d\alpha = \frac{3-z}{2}.$$



8. (i)

$$\begin{aligned} F_Z(z|Y=y) &= P[Z \leq z|Y=y] \\ &= P[X/Y \leq z|Y=y] \\ &= P[X \leq zy|Y=y] \\ &= P[X \leq zy] \\ &= F_X(zy) \end{aligned}$$

$$\begin{aligned} f_Z(z|Y=y) &= y f_X(zy) \\ &= y(zy)e^{-\frac{(zy)^2}{2}} \\ &= zy^2 e^{-\frac{z^2y^2}{2}} \quad \text{for} \quad z \geq 0. \end{aligned}$$

(ii)

$$\begin{aligned} f_Z(z) &= \int_0^\infty f_Z(z|Y=y) f_Y(y) dy \\ &= \int_0^\infty zy^2 e^{-\frac{z^2y^2}{2}} y e^{-\frac{y^2}{2}} dy \\ &= z\int_0^\infty y^3 e^{-y^2 \frac{z^2+1}{2}} dy \\ &= \frac{z}{2\left(\frac{z^2+1}{2}\right)^2} \\ &= \frac{2z}{(z^2+1)^2} \quad \text{for} \quad z \geq 0. \end{aligned}$$

9. Since X and Y are i. i. d. zero-mean Gaussian random variables with variance σ^2, we have

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

The Jacobian for the transformation $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$ is

$$J(x,y) = \det\left(\begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix}\right) = \det\left(\begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix}\right) = \frac{1}{\sqrt{x^2+y^2}}$$

The inverse transformation is $x = r\cos\theta$ and $y = r\sin\theta$.

$$f_{R,\Theta}(r,\theta) = \frac{f_{X,Y}(r\cos\theta, r\sin\theta)}{|J(r\cos\theta, r\sin\theta)|}$$

Therefore, we have

$$f_{R,\Theta}(r,\theta) = \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}}$$

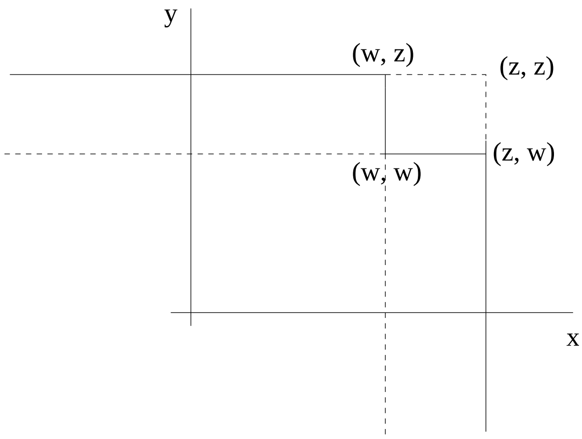
for $0 \leq r < \infty$ and $0 \leq \theta \leq 2\pi$. From this, we get

$$f_R(r) = \int_0^{2\pi} f_{R,\Theta}(r,\theta) d\theta = \int_0^{2\pi} \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} d\theta = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}$$

for $0 \leq r < \infty$. Similarly, we get

$$f_\Theta(\theta) = \int_0^\infty f_{R,\Theta}(r,\theta) dr = \int_0^\infty \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr = \frac{1}{2\pi}$$

for $0 \leq \theta \leq 2\pi$. Now, we see that $f_{R,\Theta}(r,\theta) = f_R(r) f_\Theta(\theta)$. Therefore, R and Θ are independent.



10.

$$F_{Z,W}(z,w) = P[\max(X,Y) \le z, \min(X,Y) \le w]$$

$$= \begin{cases} F_{X,Y}(w,z) + F_{X,Y}(z,w) - F_{X,Y}(w,w) & z \ge w \\ 0 & z < w \end{cases}$$

Therefore,

$$f_{Z,W}(z,w) = \begin{cases} f_{X,Y}(w,z) + f_{X,Y}(z,w) & z \ge w \\ 0 & z < w \end{cases}$$

When X and Y are i.i.d. and

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & \text{else} \end{cases}$$

we have

$$f_{X,Y}(x,y) = \lambda^2 e^{-\lambda(x+y)}$$

and

$$f_{Z,W}(z,w) = \begin{cases} 2\lambda^2 e^{-\lambda(w+z)} & z \ge w \\ 0 & \text{else} \end{cases}$$

11. We have $Z = \sqrt{X^2+Y^2}$ and $W = X/Y$. Solving for X and Y, we get $X = YW$ and

$$Z = \sqrt{Y^2W^2 + Y^2} = \pm Y\sqrt{W^2+1}.$$

Therefore, we have the following two solutions for X and Y:

(i) $Y = \frac{Z}{\sqrt{W^2+1}}, X = \frac{ZW}{\sqrt{W^2+1}}$, and
(ii) $Y = -\frac{Z}{\sqrt{W^2+1}}, X = -\frac{ZW}{\sqrt{W^2+1}}$.

Thus, an infinitesimal rectangle in the $zw-$plane corresponds to two infinitesimal regions in the $xy-$plane. The determinant of the Jacobian for both these regions is identical and equal to

$$\det\left(\begin{bmatrix} \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{w^2+1}}\right) & \frac{\partial}{\partial w}\left(\frac{z}{\sqrt{w^2+1}}\right) \\ \frac{\partial}{\partial z}\left(\frac{zw}{\sqrt{w^2+1}}\right) & \frac{\partial}{\partial w}\left(\frac{zw}{\sqrt{w^2+1}}\right) \end{bmatrix}\right) = \det\left(\begin{bmatrix} \frac{1}{\sqrt{w^2+1}} & -\frac{zw}{(w^2+1)^{3/2}} \\ \frac{w}{\sqrt{w^2+1}} & \left(-\frac{zw^2}{(w^2+1)^{3/2}} + \frac{z}{\sqrt{w^2+1}}\right) \end{bmatrix}\right).$$

Therefore, we have

$$|J| = \frac{z}{w^2+1}.$$

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}.$$

$$f_{Z,W}(z,w) = |J| f_{X,Y}\left(\frac{zw}{\sqrt{w^2+1}}, \frac{z}{\sqrt{w^2+1}}\right) + |J| f_{X,Y}\left(-\frac{zw}{\sqrt{w^2+1}}, -\frac{z}{\sqrt{w^2+1}}\right).$$

$$f_{Z,W}(z,w) = \frac{2}{2\pi}\frac{z}{w^2+1} e^{-\frac{z^2}{2}} = \frac{1}{\pi(w^2+1)} z e^{-\frac{z^2}{2}} \quad \text{for} \quad z \geq 0.$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{Z,W}(z,w)dw = z e^{-\frac{z^2}{2}} \quad \text{for} \quad z \geq 0.$$

$$f_W(w) = \int_0^{\infty} f_{Z,W}(z,w)dz = \frac{1}{\pi(w^2+1)}.$$

Z is Rayleigh distributed and W is Cauchy distributed.