EE 511 Problem Set 4

1. Consider a Gaussian random variable X with $E[X] = 0$ and $var(X) = 1$. (i) Determine the moment generating function $\phi_X(s)$, and (ii) Using the result of part (i), determine the Chernoff bound for $P[X \geq a]$. (iii) Use the Chebyshev inequality $P[|X - E[X]| \geq \delta] \leq var(X)/\delta^2$, to derive another bound for $P[X \geq a]$.

2. Consider two zero-mean random variables X and Y. Let $Z = X + aY$. Suppose that Z is independent of Y. Show that $E[X|Y = y] = -ay$.

3. Let X be a uniform random variable in $[0, 100]$. Determine $E[X]$ and $E[X|X \geq 65]$.

4. Let X be a Poisson random variable with probability mass function

$$P_X(k) = \frac{e^{-a}a^k}{k!} \quad \text{for} \quad k = 0, 1, 2, \cdots.$$

   Determine $E[X]$ and $Var(X)$.

5. Consider the random variable X with pdf

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{else} \end{cases}$$

   Determine E[X], $f_X(x|X \geq 2)$, and $E[X|X \geq 2]$.

6. Let Y be a random variable. (i) We want to choose a constant c as an estimate of Y such that $E[(Y - c)^2]$ (mean squared error) is minimized. Prove that the minimum mean squared error (MMSE) constant estimate of Y is $c = E[Y]$ and that the MMSE is $var(Y)$. (ii) Now, suppose that we can observe the random variable X and we want to choose a function $g(.)$ such that $E[(Y - g(X))^2]$ is minimized. Write $E[(Y - g(X))^2]$ in terms of $E[(Y - g(X))^2|X = x]$ and show that the optimal solution for $g(x)$ is the conditional mean of Y given $X = x$, i.e., $g(x) = E[Y|X = x]$.

7. Let X be a zero-mean Gaussian random variable with variance σ^2. Determine $E[X^n]$ for $n = 2, 3, \cdots$ in terms of σ.

8. $\underline{X} = [X_1 \;\; X_2 \;\; X_3]^T$ is a three-dimensional zero-mean Gaussian random vector with covariance matrix C given by

$$C = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

   a) Give an expression for $f_{\underline{X}}(\underline{x})$. b) If $Y = X_1 + 2X_2 - X_3$, determine $f_Y(y)$. c) Determine $f_{\underline{Z}}(\underline{z})$ for the following transformation:

$$\underline{Z} = \begin{bmatrix} 5 & -3 & -1 \\ -1 & 3 & -1 \\ 1 & 0 & 1 \end{bmatrix} \underline{X}$$

9. Let $\underline{X} = [X_1 \;\; X_2]^T$ be a two-dimensional zero-mean Gaussian random vector with covariance matrix C given by
$$C = \begin{bmatrix} 1 & r \\ r & 2 \end{bmatrix}.$$
(a) Give an expression for $f_{X_2}(x_2)$.
(b) Determine the conditional pdf, conditional mean and conditional variance of X_1 given $X_2 = x_2$.

10. (a) If X_1 and X_2 are jointly Gaussian random variables, show that their marginal distributions are also Gaussian. (b) If X_1 and X_2 are jointly distributed as
$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi} \exp\left\{-\frac{1}{2}(x_1^2 + x_2^2)\right\} \left(1 + x_1 x_2 \exp\left\{-\frac{1}{2}(x_1^2 + x_2^2 - 2)\right\}\right),$$
determine the marginal pdf's of X_1 and X_2.

11. Given random vectors $\underline{X}$ and $\underline{Y} = A\underline{X} + \underline{b}$, express $E[\underline{Y}]$ and the covariance matrix of $\underline{Y}$ in terms of $E[\underline{X}]$ and the covariance matrix of $\underline{X}$.

12. A complex random vector $\underline{X} = \underline{X_r} + j\underline{X_i}$ is proper if all elements of its pseudo-covariance matrix $E[(\underline{X} - E[\underline{X}])(\underline{X} - E[\underline{X}])^T]$ are zero. Express this condition in terms of the covariance and cross-covariance matrices of $\underline{X_r}$ and $\underline{X_i}$.

13. If $\underline{X}$ is a jointly Gaussian real random vector with
$$\phi_{\underline{X}}(\underline{s}) = \exp\left\{\underline{s}^T \underline{m} + \frac{1}{2}\underline{s}^T C \underline{s}\right\},$$
show that $E[\underline{X}] = \underline{m}$ and $E[(\underline{X} - E[\underline{X}])(\underline{X} - E[\underline{X}])^T] = C$.

14. X_1 and X_2 are jointly Gaussian with zero-mean and covariance matrix
$$C = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}.$$
Show that $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$ are uncorrelated and independent.

15. X_1 and X_2 are jointly Gaussian with zero-mean and covariance matrix
$$C = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$
Show that
$$Y_1 = \frac{X_1}{\sigma_1} + \frac{X_2}{\sigma_2} \quad \text{and} \quad Y_2 = \frac{X_1}{\sigma_1} - \frac{X_2}{\sigma_2}$$
are independent.